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Unlocking the axion-dilaton in 5D supergravity

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ABSTRACT: We revisit supersymmetric solutions to five dimensional ungauged N=1 supergravity with dynamic hypermultiplets. In particular we focus on a truncation to the axion-dilaton contained in the universal hypermultiplet. The relevant solutions are fibrations over a four-dimensional Kähler base with a holomorphic axion-dilaton. We focus on solutions with additional symmetries and classify Killing vectors which preserve the additional structure imposed by supersymmetry; in particular we extend the existing classification of solutions with a space-like U(1) isometry to the case where the Killing vector is rotational. We elaborate on general geometrical aspects which we illustrate in some simple examples. We especially discuss solutions describing the backreaction of M2-branes, which for example play a role in the black hole deconstruction proposal for microstate geometries.

KEYWORDS: Supergravity Models, D-branes, M-Theory, Black Holes in String Theory

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1 Introduction

Supergravity in 5 dimensions with 8 real supercharges plays an important role in our explorations of (quantum) gravity. Through type IIA/M-theory duality compactified on a Calabi-Yau threefold we can think of it as describing a strong coupling regime of 4 dimensional (super)gravity. One particular example where this is of use is in studying 4 dimensional extremal black holes, which were given a first quantum mechanical interpretation by Maldacena, Strominger and Witten [1] using this connection between 4d/5d supergravity. More generally this is but one aspect of a particular $\text{AdS}_3/\text{CFT}_2$ correspondence, based on the maximally supersymmetric $\text{AdS}_3 \times S^2$ vacuum solution to $N = 1$ 5D supergravity.

Supersymmetric solutions to the theory have been much studied, and actually formally classified. The first step was made in the classic work [2], where the case of ‘pure’ supergravity, i.e. the theory containing only the gravity multiplet, was analyzed. More recently this was extended to theories containing an arbitrary number of vector- and hypermultiplets in [3]. These general analyses are quite powerful and elegant, as they manage to simplify the Killing spinor equations into a small number of essentially geometric conditions. However in the presence of hypermultiplets one of these conditions, the requirement that the hyperscalars form a ‘quaternionic map’ [3] is still quite hard to explicitly address, and leads to non-trivial and complicated consequences for the underlying geometry. These have only been worked out in a small number of explicit examples, including [3–7].

In this work we progress towards a more explicit understanding and formulation of such solutions with dynamical (i.e. non-constant) hypermultiplets. The main simplification that allows us to move forward is to make a consistent truncation in the hypermultiplet sector to the axion-dilaton part of the universal hypermultiplet. In doing so we will review parts of the general story, pointing out a few observations that went unnoticed in the literature.

Before we give a summary and overview of the paper, let us shortly mention the particular puzzle that motivated this study. In [8], a proposal was made for the brane configuration representing a typical microstate geometry of the four dimensional D4/D0 black hole in type IIA compactified on a Calabi-Yau threefold. The configuration consists of a wrapped D6-brane and anti-D6 brane with worldvolume fluxes, surrounded by an ellipsoidal D2 brane. When lifted to five dimensional supergravity describing M theory on a Calabi-Yau, this configuration becomes a two-centered Taub-NUT system with an M2-brane which sources the axion-dilaton. In [9] this configuration was argued to fit within a certain ansatz for the metric and other fields. However one can verify that this ansatz is incompatible with the classification of solutions with a spacelike isometry in [3]. The resolution to this puzzle is, as we will see, is that in [3] the solution was assumed to be invariant under an isometry of the translational type, while the brane configuration of interest is instead invariant under an isometry of the rotational type. We therefore need to generalize the analysis of [3] to the case of a rotational Killing vector, and we will indeed find that the results are compatible with the ansatz of [9].

1.1 Summary and overview

In section 2 we first review the structure of supersymmetric solutions to ungauged 5D $N=1$ supergravity following [3]. Such solutions always have at least one Killing vector and we

will focus on the case where this Killing vector is time-like. The metric can then be written as a time-like fibration over a 4D Euclidean manifold that is referred to as the *base*. When the hypermultiplets are constant this base is hyperkähler, and when they are dynamical the base is more exotic, essentially almost quaternionic where the quaternionic structure is covariantly constant with respect to a scalar dependent connection. The hypermultiplets themselves are restricted to form a ‘quaternionic map’.

Inside the universal hypermultiplet, whose target space is $\frac{\text{SU}(1,2)}{\text{U}(2)}$, there sits a complex scalar τ that parameterizes the subspace $\frac{\text{SU}(1,1)}{\text{U}(1)}$. We will refer to τ as the *axidilaton*, and its real and imaginary part can be identified with the Hodge dual of the totally external part of the M-theory 4-form and the volume modulus of the Calabi-Yau respectively. The theory can be consistently truncated keeping τ as the only dynamical scalar in the hypermultiplet sector. Our main result of section 2 is the analysis of the conditions for supersymmetry in this truncated theory. We show that the base is a Kähler manifold and that the ‘quaternionic map’ condition simply becomes the requirement that τ be holomorphic. The backreaction of the axidilaton on the geometry is encoded in a differential equation for the Kähler potential, where the standard Monge-Ampère equation describing hyperkähler geometry is now deformed by a source term depending on the the axidilaton, see (2.32) below.

Even when all hypermultiplets are trivial and the base is hyperkähler, not much is known about that base in the general case without isometries. So to make progress, also when hypermultiplets are present, it will be useful to study the case where there is an additional symmetry on the base. Most solutions relevant for applications possess extra symmetries, and especially if one wants to make contact with 4D supergravity demanding a space-like $\text{U}(1)$ isometry is natural.

In section 3 we work out how the presence of one or two Killing vectors further simplifies the solutions. We begin by classifying the isometries which preserve the additional structures imposed by supersymmetry. As in the hyperkähler case [10, 11] one can distinguish between holomorphic isometries of the ‘translational’ and ‘rotational’ type depending on whether or not they preserve the almost quaternionic structure. When the axidilaton is turned on, there is a further distinction which arises from the $\text{SU}(1,1)$ U-duality of the theory. Indeed, in order to have a symmetry it is sufficient for the axidilaton to be left invariant modulo a U-duality transformation when transported along the Killing vector. There are essentially four cases — either τ is invariant (*I*) under such transport, or it transforms with a parabolic (*P*), elliptic (*E*) or hyperbolic (*H*) U-duality — and we summarized how they constrain the functional form of τ in (3.4).

The simplest case (*I*), when τ is left invariant by the Killing vector, is the most straightforward and we discuss it in depth in section 3.2. We show how the sourced Monge-Ampère equation reduces to a sourced $\text{SU}(\infty)$ Toda equation in the rotational case and to a sourced flat Laplace equation in the translational case. We also point out how one can simplify the remaining equations for the vector multiplets and highlight the rotational case, extending the analysis of [22] in the absence of hypermultiplets. To our surprise certain features, like the presence of a simple algebraic stability bound on the location of

charge centers/branes, remain intact in the more involved solutions with a rotational Killing vector. In the last part of section 3.2 we use an ansatz based on separation of variables that reduces the Toda equation underlying the geometry to a simpler sourced Liouville equation. This ansatz contains solutions with a larger, non-abelian group of symmetries for the supergravity background, and we provide explicit solutions to all 5d fields, including all vector multiplets, up to a single function obeying the sourced Liouville equation.

Having finished the discussion of a single Killing vector we move on to discuss the case of two commuting Killing vectors in section 3.3, focusing on the situation where the Killing vectors are Hamiltonian and the geometry of the base is toric. Analyzing the additional Killing vector along the lines of section 3.1 one finds that one of the two Killing vectors will always leave the axidilaton τ invariant (which explains our focus on that case in section 3.2) so that the pair of Killing vectors can be of type *II*, *IP*, *IH* or *IE*. Furthermore, the space-time dependence of the scalar τ is completely fixed, up to some integration constants and a discrete choice related to the type of symmetries. In particular, in class *II* the axidilaton is forced to be constant, and as a warm-up we first review this case in our formalism. We focus on the case where one of the Killing vectors is translational and the other is rotational and show how the solutions to the Toda equation as well to the equations governing all 5D fields, are reduced to specifying a number of axially symmetric harmonic functions in \mathbb{R}^3 . We then turn to the situation with nonconstant dilaton. In order to have a dynamical axidilaton in the presence of two commuting symmetries, one is forced to have τ transform with a non-trivial U-duality under one of the symmetries. Furthermore if that symmetry is along a compact direction it means τ will have monodromy, which signals the presence of brane sources. In section 3.3.3 we discuss in more detail how and under which conditions the solutions with toric symmetry can be interpreted as backreacted M2 branes that are extended in the 5d external directions. Finally, in section 3.3.4, we analyze the special class of separable solutions to the Toda equation under the additional assumption of toric symmetry. This concludes our general analysis of dynamic axidilaton solutions.

After our rather abstract and technical discussion in the first sections we illustrate all the features discussed there in a number of examples in section 4. We start by reviewing some physically interesting solutions with a toric hyperkähler base in our formalism, including cases where the base is ‘ambipolar’ and changes signature in some region, commenting on the generalization of toric geometry which governs these spaces. We then turn to solutions with axidilaton, focussing on those solutions which describe backreacted M2-branes placed in a background with a toric base. We discuss in detail backreacted branes in flat space and the highly symmetric Gödel \times S² solution [9] which, as we will argue, arises from a distribution of branes in the AdS₃ \times S² background. We also comment on the solutions describing individual branes in the Eguchi-Hanson, AdS₃ \times S², and AdS₂ \times S³ backgrounds, the latter being of interest for the black hole deconstruction proposal [8]. These solutions are fully specified by a single function satisfying an ordinary non-linear differential equation. They will be discussed in more detail using the tools developed in this paper in a future publication [12].

2 Characterizing supersymmetric 5D axidilaton solutions

In this section we first review some basics of 5D N=1 supergravity and its relation to M-theory compactified on a Calabi-Yau manifold. We point out some of the essential equations and geometric structure that govern general supersymmetry preserving solutions to this theory, following the work of [3]. We analyze this general structure in more depth in the case of truncation to the axion-dilaton scalars inside the universal hypermultiplet and show how solutions are completely determined by a choice of holomorphic axion-dilaton profile and a single remaining complex non-linear equation, essentially a sourced Monge-Ampère equation.

2.1 N=1 supergravity from M-theory on a Calabi-Yau

Local supersymmetry in 4+1 dimensions requires a minimum of 8 real supercharges, which we will call 5D N=1 supersymmetry. In this work we will consider ungauged N=1 supergravity theories that apart from the gravity multiplet contain couplings to vector multiplets and hypermultiplets. Let us briefly review the bosonic field content and the geometry governing such a theory. The bosonic fields of the gravity multiplet are the metric and the graviphoton A^0 . Each of the n_v vector multiplets contains a massless vector A^x and a real scalar ϕ^x , $x = 1, \dots, n_v$. The vector multiplet sector is governed by very special real geometry [13, 14]. The matter vectors A^x can be combined with A^0 into a column vector A^I , $I = 0, \dots, n_V$ transforming as a vector under an $SO(n_v + 1)$ global symmetry of the theory. Similarly it is convenient to describe the scalar manifold in terms of $n_v + 1$ homogeneous coordinates $Y^I(\phi)$ satisfying a constraint

$$D_{IJK}Y^IY^JY^K = 6. \quad (2.1)$$

Here, D_{IJK} is a totally symmetric $SO(n_v+1)$ tensor which completely determines the metric $g_{xy}(\phi)$ on the scalar manifold and the scalar-dependent kinetic term $a_{IJ}(\phi)$ for the vectors. For explicit expressions we refer to [3], appendix A.3.¹ Each of the n_h hypermultiplets contains 4 real scalars which we collectively denote as q^X , $X = 1, \dots, 4n_h$, whose target space is a quaternionic Kähler manifold [15] with metric $g_{XY}(q)$.

The bosonic part of the most general 2-derivative supersymmetric Lagrangian describing these fields is

$$S = \int d^5x \sqrt{-g} \left[R - \frac{1}{2} g_{xy}(\phi) \partial_\mu \phi^x \partial^\mu \phi^y - \frac{1}{2} g_{XY}(q) \partial_\mu q^X \partial^\mu q^Y - \frac{1}{4} a_{IJ}(\phi) F_{\mu\nu}^I F^{J\mu\nu} \right] + \frac{D_{IJK}}{6} \int F^I \wedge F^J \wedge A^K. \quad (2.2)$$

We will be especially interested in the 4+1 dimensional theory arising from compactifying 11-dimensional supergravity on a Calabi-Yau manifold X [16]. The field content of the 5D theory is now directly related to the Hodge-numbers $h_{(i,j)}$ of X . Besides the gravity multiplet, this theory contains $h_{(1,1)} - 1$ vector multiplets where the tensor D_{IJK}

¹We follow essentially the conventions of [3], with exception of the metric signature (ours is mostly plus), and the quantities h^I and C_{IJK} in [3] are related to ours as $h^I = Y^I/\sqrt{3}$, $C_{IJK} = \sqrt{3}D_{IJK}/2$.

determining the real geometry is given by the intersection matrix on X . The hypermultiplets consist of the universal hypermultiplet [17], whose couplings are independent of the topology of X , and an additional $h_{(2,1)}$ hypermultiplets. In this work we will consider solutions where only the universal hypermultiplet plays a role. Its bosonic fields, viewed as 2 complex scalars, arise as follows. The first one of these is an axion-dilaton-like field, with a real part which is the Hodge dual of the three form with legs in the 5D spacetime, and an imaginary part coming from the volume modulus of X . We will refer to this field as the axidilaton τ . The other complex scalar arises from the three form modes proportional to the $(3,0)$ and $(0,3)$ form on X . The hypermultiplet moduli space is a direct product of the universal hypermultiplet moduli space and that of the remaining $h_{(2,1)}$ hypermultiplets. The universal part of the hypermultiplet moduli space is the homogeneous quaternionic space $\frac{\text{SU}(1,2)}{\text{U}(2)}$.

2.2 Structure of supersymmetric solutions

In [3], the general structure of supersymmetric solutions of 4+1 dimensional supergravity with vector- and hypermultiplets was analyzed, extending the pioneering work on minimal supergravity in [2]. The idea is to assume the existence of a Killing spinor and analyze how the Killing spinor equations constrain the bosonic fields constructed out of Killing spinor bilinears. We now briefly review the results of this analysis. A first spinor bilinear yields a Killing vector, which in the current work will be assumed to be everywhere timelike. Choosing an adapted coordinate, the metric is of the form

$$ds^2 = -f^2(dt + \xi)^2 + f^{-1}ds_4^2 \quad (2.3)$$

where ds_4^2 denotes the Euclidean metric on a 4-dimensional base manifold which we will refer to as the *base*.

Let's first discuss the BPS equations which constrain the geometry of the base, which do not involve the vector multiplets. There exist three selfdual 2-forms Φ^a , $a = 1, 2, 3$ which endow the base with an almost quaternionic structure:

$$\Phi^a = \star_4 \Phi^a \quad (2.4)$$

$$\Phi^a{}^A{}_C \Phi^b{}^C{}_B = -\delta^{ab} \delta_B^A + \epsilon^{ab}{}_c \Phi^c{}^A{}_B. \quad (2.5)$$

Our index convention is as follows: $A, B = 1, \dots, 4$ are 4D tangent space indices, while 4D curved indices will be denoted by $\mu, \nu = 1, \dots, 4$. Note that these relations are invariant under local $\text{SO}(3)$ transformations under which the Φ^a transform as a triplet. The BPS equations governing these two-forms are

$$\nabla_\mu \Phi^a{}_{BC} + \epsilon^a{}_{bc} A_\mu^b \Phi^c{}_{BC} = 0. \quad (2.6)$$

With a slight abuse of notation we have denoted by $A^a \equiv A_X^a dq^X$ the pullback of the $\text{SU}(2) \subset \text{SO}(4)$ part of the spin connection on the quaternionic hypermanifold. We note that, when the hyperscalars are constant, this equation tells us that the Φ^a must be co-variantly constant and hence endow the base with a hyperkähler structure.

The hypermultiplet scalars q^X parameterize a map from the base into the quaternionic target space which is constrained by the BPS condition

$$(dq^X)_A = \Phi^a{}_A{}^B (dq^Y)_B J^a{}_Y{}^X \quad (2.7)$$

where the $J^a{}_Y{}^X$ form the quaternionic structure of the hypermultiplet target space. This type of map was called a *quaternionic map* in [3]. One of the purposes of this work is to demystify this condition in the simplest context when only the axidilaton is turned on, where we will see that it reduces to a simple holomorphicity condition.

In addition to equations (2.6), (2.7), there are additional BPS conditions which determine the warp factor f and the one-form ξ in (2.3), as well as the vector multiplet scalars Y^I and the Maxwell field strengths F^I . Supersymmetry relates all of these fields to $n_V + 1$ harmonic anti-selfdual 2-forms Θ^I on the 4D base as follows:

$$d\Theta^I = 0, \quad \star_4 \Theta^I = -\Theta^I \quad (2.8)$$

$$\nabla_4^2(f^{-1}Y_I) = \frac{1}{2}D_{IJK}\Theta^J \cdot \Theta^K \quad (2.9)$$

$$d\xi - \star_4 d\xi = \frac{1}{2}f^{-1}Y_I\Theta^I \quad (2.10)$$

$$F^I = -d(fY^I(dt + \xi)) + \Theta^I \quad (2.11)$$

where $Y_I \equiv D_{IJK}Y^JY^K$, $\alpha \cdot \beta = \alpha_{\mu\nu}\beta^{\mu\nu}$ and the vector multiplet scalars Y^I satisfy the constraint $D_{IJK}Y^IY^JY^K = 6$.

2.3 Axidilaton solutions

In the 5D supergravity theory arising from 11-dimensional supergravity on a Calabi-Yau manifold, the hypermultiplet moduli space is a direct product of a universal hypermultiplet component and a component associated to the remaining $h_{(2,1)}$ hypermultiplets. The theory therefore allows a consistent truncation to the class of solutions where only the universal hypermultiplet is turned on. In that case the 4-dimensional hypermultiplet moduli space is $\frac{\text{SU}(1,2)}{\text{U}(2)}$, see appendix A for a brief review and conventions. The moduli space has an $\text{SU}(1,2)$ isometry which acts as a U-duality group of the 5D fields. Note that the two-forms Φ^a also transform under U-duality. Indeed, from (2.6) we see that the Φ^a not only transform as two-forms under diffeomorphisms, but also rotate into each other under local frame rotations of the hypermultiplet target space. This determines how the Φ^a transform under U-duality: an $\text{SU}(1,2)$ U-duality induces an $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)'$ frame rotation in target space, and the Φ^a rotate into each other under the pullback of the $\text{SU}(2)$ part. The metric on the 4D base is however U-duality invariant.

Now we consider hypermultiplet solutions where only the axidilaton is turned on. This means we look at a further consistent truncation of the theory where two of the scalars q_3, q_4 are constant while q_1, q_2 can fluctuate. Without loss of generality, we will set $q_3 = q_4 = 0$ in what follows. We use the standard notation $\tau = \tau_1 + i\tau_2$ for the axidilaton, with

$$q^1 = -\tau_2, \quad q^2 = -\tau_1. \quad (2.12)$$

The hyperscalar metric (A.3) on this submanifold is

$$ds^2 = \frac{d\tau d\bar{\tau}}{4\tau_2^2}. \quad (2.13)$$

The part of the original U-duality group which leaves the subspace $q_3 = q_4 = 0$ invariant is $SU(1, 1) \simeq SL(2, \mathbb{R})$, which acts on τ as the familiar fractional linear transformations.

The $SU(2)$ connection A^a , which in our conventions is given by (A.8), becomes

$$A^1 = A^2 = 0 \quad (2.14)$$

$$A^3 = -\frac{d\tau_1}{2\tau_2}. \quad (2.15)$$

For pure axidilaton solutions eqs. (2.6) simplify to, defining $\Phi^\pm = \Phi^1 \pm i\Phi^2$:

$$\nabla_\mu \Phi^\pm \pm iA_\mu^3 \Phi^\pm = 0 \quad (2.16)$$

$$\nabla_\mu \Phi^3 = 0 \quad (2.17)$$

The last equation states that the almost complex structure Φ^3 is covariantly constant. Hence when turning on only the axidilaton, the base is still Kähler, with Kähler form Φ^3 , but it will in general no longer be hyperkähler. Note that the completely antisymmetric part of (2.16) can be written as

$$d\Phi^\pm \pm iA^3 \wedge \Phi^\pm = 0. \quad (2.18)$$

As discussed above, the forms Φ^a transform under U-dualities. Since A^3 transforms under fractional linear transformations as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad A_3 \rightarrow A_3 - d\text{Im} \log(c\tau + d) \quad (2.19)$$

the two-forms Φ^\pm must transform by a phase in order for (2.16) to remain invariant:

$$\Phi^\pm \rightarrow e^{\pm i\text{Im} \log(c\tau + d)} \Phi^\pm. \quad (2.20)$$

Exploiting the fact that the metric is Kähler with respect to Φ^3 , we introduce adapted complex coordinates w^1, w^2 as well as a unitary frame φ^1, φ^2 of $(1, 0)$ forms such that

$$ds^2 = g_{i\bar{j}} dw^i d\bar{w}^j = \varphi^1 \bar{\varphi}^1 + \varphi^2 \bar{\varphi}^2 \quad (2.21)$$

$$\Phi^3 = \frac{i}{2} g_{i\bar{j}} dw^i \wedge d\bar{w}^j = -i (\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2). \quad (2.22)$$

It follows from (2.5) that Φ^+ and Φ^- are of type $(2, 0)$ and $(0, 2)$ respectively, and that $\Phi^+ \wedge \Phi^- = 2\Phi^3 \wedge \Phi^3$. Using also that $\Phi^- = \overline{\Phi^+}$ fixes Φ^+ up to a real phase λ

$$\Phi^+ = e^{i\lambda} \varphi^1 \wedge \varphi^2. \quad (2.23)$$

(The phase λ could be absorbed by a frame rotation, but we prefer to keep our frame arbitrary.)

First, let's analyze the equations (2.7) for the axidilaton in this frame. Using (A.7) one finds that they are equivalent to

$$(d\tau)_{\bar{\varphi}^1} = (d\tau)_{\bar{\varphi}^2} = 0 \quad (2.24)$$

where the subscripts denote components in the unitary basis (2.22). In other words, the quaternionic map condition (2.7) here simply states that τ must be a holomorphic function:

$$\partial_{\bar{w}^1}\tau = \partial_{\bar{w}^2}\tau = 0. \quad (2.25)$$

Now we turn to the antisymmetrized equations (2.18) for Φ^\pm which reduce to

$$d\Phi^+ - i\frac{d\tau_1}{2\tau_2} \wedge \Phi^+ = 0 \quad (2.26)$$

and the complex conjugate thereof. Arguments similar to the one above (2.23) show that (2.5) determines Φ^+ in the coordinate basis up to a real phase α :

$$\Phi^+ = \sqrt{g_{\mathbb{C}}} e^{i\alpha} dw^1 \wedge dw^2. \quad (2.27)$$

where $g_{\mathbb{C}} \equiv \det\{g_{i\bar{j}}\} = \sqrt{\det g}$. Using that $d = \partial + \bar{\partial}$ and the holomorphicity of τ one finds that (2.26) is equivalent to

$$\bar{\partial} \log \frac{g_{\mathbb{C}} e^{2i\alpha}}{\tau_2} = 0 \quad (2.28)$$

This implies that there exists a holomorphic function $h(w^1, w^2)$ such that

$$\frac{g_{\mathbb{C}} e^{2i\alpha}}{\tau_2} = e^h \quad (2.29)$$

Furthermore, since both $g_{\mathbb{C}}$ and τ_2 are strictly positive and α is real it follows, setting $h \equiv h_1 + ih_2$, that

$$h_2 = 2\alpha \quad (2.30)$$

Hence we obtain the following constraint on the base metric:

$$g_{\mathbb{C}} = \tau_2 e^{h_1}. \quad (2.31)$$

In summary, a general supersymmetric solution is specified by two holomorphic functions τ, h and a metric which is Kähler and satisfies (2.31). The latter two conditions can be combined into a nonlinear differential equation for the Kähler potential, which for later convenience we normalize as $g_{i\bar{j}} = 4\mathcal{K}_{i\bar{j}}$,

$$\mathcal{K}_{1\bar{1}}\mathcal{K}_{2\bar{2}} - \mathcal{K}_{1\bar{2}}\mathcal{K}_{2\bar{1}} = \frac{\tau_2 e^{h_1}}{16}. \quad (2.32)$$

This is a nonlinear partial differential equation of the Monge-Ampère type, see e.g. [19]. In the case of constant axidilaton, it is the familiar Monge-Ampère equation expressing that the Kähler base is Ricci flat, which is equivalent to the hyperkähler condition in 4 real dimensions. A non-constant axidilaton backreacts on the metric by introducing a source in

the r.h.s. of the equation and deforming the geometry away from being hyperkähler. The Ricci tensor of the base is

$$R_{i\bar{j}} = -i\partial_i\partial_{\bar{j}} \ln g_{\mathbb{C}} = -i\partial_i\partial_{\bar{j}} \ln \tau_2 \quad (2.33)$$

and the two-forms Φ^+ are given by

$$\Phi^+ = \frac{\sqrt{\tau_2} e^{\frac{h}{2}}}{4} dw^1 \wedge dw^2, \quad \Phi^- = \overline{\Phi^+} \quad (2.34)$$

Although so far we have only imposed the fully antisymmetric part (2.18) of the equations (2.16), we have checked that the remaining equations in (2.16) are automatically satisfied.

So far we have not yet chosen specific holomorphic coordinates and we are free to make holomorphic coordinate transformations. We should note however that, while τ and \mathcal{K} transform as scalars, h must transform nontrivially such that eq. (2.32) is invariant. In particular, e^h must be a density of weight 2 so that under $w \rightarrow \tilde{w}(w)$ the field h transforms as

$$\tilde{h}(\tilde{w}) = h(\tilde{w}) + 2 \ln \det \left(\frac{\partial \tilde{w}^i}{\partial w^j} \right) \quad (2.35)$$

The infinitesimal transformation of h generated by a holomorphic vector field k is by definition the Lie derivative, which has an extra term compared to the Lie derivative of a scalar field:

$$\delta_k h \equiv \mathcal{L}_k h = k^i h_{,i} + 2\partial_i k^i \quad (2.36)$$

Note that in principle this implies one can always (locally) choose coordinates in which h becomes trivial. But as will become clear in the following such coordinate choice makes other aspects of the solutions less transparent and so we prefer to keep h free and preserve manifest holomorphic coordinate invariance for now.

2.4 Redundancies

Recapitulating, we have described the configuration space of supersymmetric axidilaton solutions in terms of two holomorphic functions τ and h (recall that the imaginary part of h is the phase of Φ^+) and a real function \mathcal{K} satisfying eq. (2.32). Our description is however redundant since the following symmetry transformations on τ, h, \mathcal{K} produce equivalent configurations:

- **U-duality transformations.** The $\text{SL}(2, \mathbb{R})$ U-duality transformations act as fractional linear transformations on τ . Since the two-forms Φ^\pm are charged under U-duality and transform as (2.20), it follows from (2.30) that e^h also has a nontrivial transformation law and is in fact a modular form of weight 2:

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d}; \quad ad - bc = 1 \quad (2.37)$$

$$\tilde{h} = h + 2 \log(c\tau + d) \quad (2.38)$$

$$\tilde{\mathcal{K}} = \mathcal{K}. \quad (2.39)$$

Note that this implies that also (2.32) is invariant. Let us also write down the infinitesimal version of this transformation law. Parameterizing a general $sl(2, \mathbb{R})$ Lie algebra element as

$$Q = rL_0 + qL_1 + pL_{-1} = \begin{pmatrix} \frac{r}{2} & p \\ -q & -\frac{r}{2} \end{pmatrix} \quad (2.40)$$

for $p, q, r \in \mathbb{R}$, we have

$$\delta_U \tau = p + r\tau + q\tau^2 \quad (2.41)$$

$$\delta_U h = -r - 2q\tau \quad (2.42)$$

$$\delta_U \mathcal{K} = 0 \quad (2.43)$$

- **Kähler transformations.** As always, the Kähler potential is only defined up to addition of the real part of a holomorphic function:

$$\delta_K \mathcal{K} = \epsilon(w) + \bar{\epsilon}(\bar{w}), \quad \delta_K \tau = \delta_K h = 0 \quad (2.44)$$

- **Global U(1) rotations of Φ^\pm .** Finally, we are free to rotate Φ^\pm by a constant phase $e^{\pm is}$, which corresponds to an imaginary shift of h :

$$\delta_{rot} h = is, \quad \delta_{rot} \tau = \delta_{rot} \mathcal{K} = 0 \quad (2.45)$$

This corresponds to rotating the Killing spinor by an overall phase.

Hence it's not quite correct to think of τ, h, \mathcal{K} as functions, rather they are sections of appropriate line bundles that can undergo transformations of the above types when going to a different coordinate patch.

In particular, when going around a closed curve, the τ and h fields can pick up a monodromy by a U-duality transformation, which signals a degeneration of the internal Calabi-Yau manifold. Recalling that τ_1 is the Hodge dual of the M-theory three-form with legs in the 5D noncompact space, it is easy to see that a monodromy $\tau \rightarrow \tau + 1$ signals the presence of an M2-brane extended in the 5D noncompact space and smeared over the internal Calabi-Yau. More general $SL(2, \mathbb{Z})$ -valued monodromies signal the presence of exotic branes which do not descend from 11D M-branes [43, 44]. Note that in the present case the exotic branes are geometric from the 5D point of view, the 5D metric being single-valued when encircling these objects.

3 Structure of solutions with extra Killing vectors

In this section we discuss the simplifications which occur in the presence of one or two additional Killing vectors which preserve the structure imposed by supersymmetry. In the case of two Killing vectors we will focus on the situation where the base has (generalized) toric geometry; the geometry of the base is then fully specified by a single function satisfying an ordinary non-linear differential equation.

3.1 Single compatible Killing vector: classification

We will from now on focus on supersymmetric solutions which admit, besides the timelike Killing vector constructed out of the Killing spinor itself, an additional Killing vector on the base. It is natural to restrict attention to Killing vectors which not only preserve the metric but also the additional structure imposed by supersymmetry discussed in the previous section. We will call such Killing vectors *compatible* with the supersymmetric structure. For example, we will consider only Killing vectors which preserve the complex structure with Kähler form Φ^3 , and hence will restrict our attention to holomorphic Killing vectors. We want the Killing vector to furthermore preserve τ, h and \mathcal{K} , which is certainly the case if these functions are strictly invariant, i.e. their appropriately defined Lie derivatives (see e.g. (2.36)) vanish. This is however too strong a requirement in view of the redundancies discussed in paragraph 2.4 above: it is sufficient if their holomorphic transformation can be compensated for by a combination of a U-duality (2.39), a Kähler transformation (2.44) and a U(1) rotation (2.45). In other words, we must have a transformation law of the form

$$\mathcal{L}_k F = \delta_U F + \delta_K F + \delta_{rot} F \quad (3.1)$$

where F stands for any of the fields τ, h, \mathcal{K} . We will now explore how this requirement constrains the fields. To simplify the discussion we choose local holomorphic coordinates w^1, w^2 such that $w^1 = x^1 + i\theta^1$ is adapted to the Killing vector k :

$$k = \partial_{\theta^1} = i(\partial_{w^1} - \partial_{\bar{w}^1}). \quad (3.2)$$

Starting with the axidilaton τ , the holomorphic reparametrization must induce an infinitesimal fractional linear transformation, $\mathcal{L}_k \tau = \delta_U \tau$, so that τ must be a solution of

$$i\partial_{w^1} \tau = p + r\tau + q\tau^2 \quad (3.3)$$

for some p, q, r . The general solution of this equation has four subcases:

$$\tau = \begin{cases} \tilde{\tau}(w^2) & \text{for } Q = 0 \\ -ipw^1 + \tilde{\tau}(w^2) & \text{for } r = q = 0, p \neq 0 \\ -\frac{p}{r} + \tilde{\tau}(w^2)e^{irw^1} & \text{for } q = 0, r \neq 0 \\ -\frac{r}{2q} - i\frac{\sqrt{\det Q}}{q} \tanh\left(\sqrt{\det Q}w^1 + i\tilde{\tau}(w^2)\right) & \text{for } q \neq 0, \det Q \neq 0 \end{cases} \quad (3.4)$$

with $\tilde{\tau}$ an arbitrary function of w^2 .

We can simplify these expressions a bit by choosing a convenient U-duality frame: under a change of U-duality frame, the $sl(2, \mathbb{R})$ element Q is conjugated by an $SL(2, \mathbb{R})$ group element, which we can use to pick a simple representative within the same conjugacy class. Conjugacy classes are labeled by the value of $\det Q = pq - r^4/4$. There are four distinct cases depending on whether τ is invariant (I) or transforms by element of an elliptic (E), hyperbolic (H) or parabolic (P) conjugacy class. The representative we will choose

Class	representative	e^Q	τ
I: invariant, $Q = 0$	$p = q = r = 0$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\tilde{\tau}(w^2)$
P: parabolic, $\det Q = 0$	$r = q = 0$	$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$	$-ipw^1 + \tilde{\tau}(w^2)$
H: hyperbolic, $\det Q < 0$	$p = q = 0$	$\begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{-\frac{r}{2}} \end{pmatrix}$	$\tilde{\tau}(w^2)e^{irw^1}$
E: elliptic, $\det Q > 0$	$r = 0, p = q$	$\begin{pmatrix} \cos q & \sin q \\ -\sin q & \cos q \end{pmatrix}$	$-i \tanh(qw^1 + i\tilde{\tau}(w^2))$

Table 1. Convenient choices of representative within each U-duality conjugacy class.

in each of these classes is shown in table 1. Note that if $k = \partial_{\theta^1}$ generates a compact U(1) isometry, in all except the invariant cases τ picks up a monodromy when circling around the U(1) direction, which signals the presence of M2 (in the parabolic case) or exotic (in the hyperbolic and elliptic cases) brane charge. Turning next to the field h , the invariance condition (3.1) says that its Lie derivative amounts to a combined U-duality and U(1) rotation, $\mathcal{L}_k h = \delta_U h + \delta_{rot} h$, leading to

$$i\partial_{w^1} h = -r - 2q\tau + is_1 \quad (3.5)$$

If $Q \neq 0$ we have (except possibly at isolated points) $\partial_{w^1} \tau \neq 0$ and (3.5) can be rewritten using (3.3) as $\partial_{w^1}(h - s_1 w^1) = \partial_{w^1}(-\ln i\partial_{w^1} \tau)$. Hence (3.5) integrates to the simple general solution

$$h = \begin{cases} -\ln(i\partial_{w^1} \tau) + s_1 w^1 + \tilde{h}(w^2) & \text{for } Q \neq 0 \\ s_1 w^1 + \tilde{h}(w^2) & \text{for } Q = 0 \end{cases} \quad (3.6)$$

with \tilde{h} another arbitrary function of w^2 . In table 2 we list the expressions for h for the four types of Killing vector as well as for the combination $\tau_2 e^{h_1}$ entering in the equation (2.32) for the Kähler potential.

Let us also introduce some commonly used terminology related to the U(1) term in the transformation of h with parameter s_1 . When s_1 is zero, the two-forms Φ^\pm are invariant under the isometry (see (2.34)) and the isometry is usually called *translational* following [10]. When s_1 is nonzero, Φ^\pm have charge $\pm s_1$ and the isometry was called *rotational* in [10]. Note that in this case we have a compact U(1) isometry. If we normalize θ^2 to have period 2π , we see from (2.34) that requiring Φ^\pm to be single-valued when going around the U(1) direction requires that s_1 is quantized in units of 2. Since Φ^\pm is constructed out of a spinor

Class	h	$\tau_2 e^{h_1}$
I	$h = s_1 w^1 + \tilde{h}(w^2)$	$\tilde{\tau}_2 e^{\tilde{h}_1 + s_1 x^1}$
P	$h = s_1 w^1 + \tilde{h}(w^2)$	$(\tilde{\tau}_2 - p x^1) e^{\tilde{h}_1 + s_1 x^1}$
H	$h = (ir + s_1) w^1 + \tilde{h}(w^2)$	$(\tilde{\tau}_2 \cos r x^1 - \tilde{\tau}_1 \sin r x^1) e^{\tilde{h}_1 + s_1 x^1}$
E	$h = 2 \ln \cosh(q w^1 + i \tilde{\tau}(w^2) + s_1 w^1 + \tilde{h}(w^2))$	$\frac{1}{2} \sinh(2 \tilde{\tau}_2 - 2 q x^1) e^{\tilde{h}_1 + s_1 x^1}$

Table 2. The functions h and $\tau_2 e^{h_1}$ for the various types of Killing vector.

bilinear we see that when $s_1/2$ is even, (resp. odd), the Killing spinor has even (resp. odd) spin structure around the U(1) direction.

Now let's turn to the Kähler potential \mathcal{K} . In principle we can only ask that \mathcal{K} is invariant up to a Kähler transformation, that is,

$$\partial_{\theta^1} \mathcal{K} = f(w^i) + \bar{f}(\bar{w}^i). \quad (3.7)$$

It is easy to see however that, by making a suitable Kähler transformation, we can make the Kähler potential locally invariant.²

$$\partial_{\theta^1} \mathcal{K} = 0. \quad (3.8)$$

Choosing an invariant representative for the Kähler potential is consistent with the Monge-Ampère equation (2.32) for \mathcal{K} since one can show that, as a consequence of (3.3), (3.5), $\tau_2 e^{h_1}$ is independent of θ^2 , as can be seen in the explicit solutions above. The Monge-Ampère equation (2.32) then reduces to

$$\mathcal{K}_{w^2 \bar{w}^2} \mathcal{K}_{x^1 x^1} - \mathcal{K}_{w^2 x^1} \mathcal{K}_{\bar{w}^2 x^1} = \frac{\tau_2 e^{h_1}}{4}. \quad (3.9)$$

Incidentally, it follows for this choice of Kähler potential representative that

$$i_{\partial_{\theta^1}} \Phi_3 = -2(\bar{\partial} \mathcal{K}_{w^1} + \partial \mathcal{K}_{\bar{w}^1}) = -d(\mathcal{K}_{w^1} + \mathcal{K}_{\bar{w}^1}) = -d\mathcal{K}_{x^1} \quad (3.10)$$

Hence, if \mathcal{K}_{x^1} extends to a globally well-defined function, the Killing vector is Hamiltonian with moment map \mathcal{K}_{x^1} .

3.2 Simplifications when τ is invariant

From now on we will focus on solutions which have an adapted (in the sense explained in section 3.1) Killing vector under which τ is invariant, in other words they belong to the class *I* of the classification above. This doesn't mean that our analysis of the other classes (which we labeled by *P*, *H* and *E*) of Killing vectors was in vain however, since in section 3.3 we will consider solutions with a second Killing vector under which τ need not be invariant. Anticipating this we will slightly change notation and switch the roles

²Obstructions can arise when trying to do this simultaneously for several noncommuting Killing vectors.

of w^1 and w^2 with respect to the previous section, i.e. the invariant Killing vector is $k = \partial_{\theta^2} = i(\partial_{w^2} - \partial_{\bar{w}^2})$, and τ and h are of the form

$$\tau = \tilde{\tau}(w^1) \quad (3.11)$$

$$h = \tilde{h}(w^1) + s_2 w^2. \quad (3.12)$$

3.2.1 The Toda frame

In the case of invariant τ we just mentioned, the equation (3.9) determining the 4D metric can be cast in a more manageable form by making a Legendre transformation. We define a potential \mathcal{V} which is the Legendre transform of \mathcal{K} with respect to x^2 :

$$\mathcal{V}(w^1, \bar{w}^1, y_2) = x^2 y_2 - \mathcal{K}; \quad y_2 = \mathcal{K}_{x^2} \quad (3.13)$$

It's useful to introduce a new special symbol Ψ for the y_2 derivative of \mathcal{V} :

$$\Psi(w^1, \bar{w}^1, y_2) \equiv \mathcal{V}_{y_2}(w^1, \bar{w}^1, y_2) = x^2(w^1, \bar{w}^1, y_2). \quad (3.14)$$

We will refer to Ψ as the *Toda potential* for reasons we will now explain. Using the Legendre transformation of second derivatives

$$\mathcal{K}_{x^2 x^2} = \frac{1}{\mathcal{V}_{y_2 y_2}} \quad (3.15)$$

$$\mathcal{K}_{w^1 x^2} = -\frac{\mathcal{V}_{w^1 y_2}}{\mathcal{V}_{y_2 y_2}} \quad (3.16)$$

$$\mathcal{K}_{w^1 \bar{w}^1} = \frac{\mathcal{V}_{w^1 y_2} \mathcal{V}_{\bar{w}^1 y_2}}{\mathcal{V}_{y_2 y_2}} - \mathcal{V}_{w^1 \bar{w}^1}. \quad (3.17)$$

we see that the Monge-Ampère equation (3.9) reduces to the following equation for Ψ ,

$$4\Psi_{w^1 \bar{w}^1} + \frac{\tilde{\tau}_2 e^{\tilde{h}_1}}{s_2} (e^{s_2 \Psi})_{y_2 y_2} = 0 \quad (3.18)$$

When the axidilaton is constant this equation reduces, after a holomorphic reparameterization setting η to zero, to the $SU(\infty)$ Toda equation or, in the limit $s_2 \rightarrow 0$, to the 3D flat Laplace equation. These are the well-known equations describing hyperkähler metrics with a rotational/translational isometry [10, 11]. See [20, 21] for more information on the $SU(\infty)$ Toda equation and its solutions.

Defining also

$$K^0 \equiv \Psi_{y_2} \quad (3.19)$$

$$\chi \equiv -2\text{Im}(\Psi_{w^1} dw^1) \quad (3.20)$$

the base metric and Kähler form are

$$ds_4^2 = K^0 ds_3^2 + \frac{1}{K^0} (d\theta^2 + \chi)^2 \quad (3.21)$$

$$ds_3^2 = dy_2^2 + \tilde{\tau}_2 e^{\tilde{h}_1} e^{s_2 \Psi} dw^1 d\bar{w}^1 \quad (3.22)$$

$$\Phi^3 = \frac{i}{2} K^0 \tilde{\tau}_2 e^{\tilde{h}_1 + s_2 \Psi} dw^1 \wedge d\bar{w}^1 + dy_2 \wedge (d\theta^2 - 2\text{Im}(\Psi_{w^1} dw^1)) \quad (3.23)$$

Note that (3.18) can be formally written as a Laplace equation $\Delta\Psi = 0$ with respect to the 3D metric (3.22), with the proviso that the 3D metric itself depends on Ψ when s_2 is nonzero. The 3D metric is not flat in general and its scalar curvature is given by

$$R^{(3)} = \frac{1}{\tilde{\tau}_2 e^{\tilde{h}_1 + s_2 \Psi}} \left(\frac{s_2}{2} (s_2 (K^0)^2 + 2K_{y_2}^0) + \frac{\tilde{\tau}' \bar{\tau}'}{\tilde{\tau}_2^2} \right). \quad (3.24)$$

3.2.2 Structure of 5D multibrane solutions

Having determined the geometry of the base in the presence of a Killing vector, we now turn to the solution of the equations (2.11) which determine the full 5D metric as well the bosonic fields in the vector multiplets. In the absence of hypermultiplets the analysis of the system of equations (2.11) on a base with rotational isometry was performed in [22] and is easily generalized to include the axidilaton. We will also expand on the discussion given there, most notably in the discussion around (3.46), (3.49). We choose to write the solution in a form which allows easy comparison with the more extensively studied solutions with translational Killing vector, to which they should reduce when the parameter s_2 is taken to zero. More details can be found in appendix B.

Starting from a solution (3.18), (3.21), (3.22) for the 4D base manifold, the general 5D solution depends on an additional set of functions K^I, K_I, K_0 and a one-form ω . Let us first discuss the algebraic structure of the general solution, which is exactly the same as for the well-known solutions with a translational Killing vector which were originally constructed as lifts of 4D solutions with vector multiplets [23]:

$$ds_5^2 = -f^2(dt + \xi)^2 + f^{-1}ds_4^2 \quad (3.25)$$

$$\Theta^I = \left(-2K^0 \star_3 d \left(\frac{K^I}{K^0} \right) \right)^- \quad (3.26)$$

$$f^{-1}Y_I = -2K_I + D_{IJK} \frac{K^J K^K}{K^0} \quad (3.27)$$

$$\xi = \frac{\omega}{2} + \frac{L}{2(K^0)^2} (d\theta^2 + \chi) \quad (3.28)$$

$$L = K_0 (K^0)^2 + \frac{1}{3} D_{IJK} K^I K^J K^K - K^I K_I K^0 \quad (3.29)$$

$$F^I = -d(fY^I(dt + \xi)) + \Theta^I \quad (3.30)$$

where we defined the (anti-) selfdual projections

$$\alpha^\pm = \frac{1}{2}(\alpha \pm \star_4 \alpha). \quad (3.31)$$

Recall that the orientation of the base was chosen such that the Kähler form is selfdual.

In order to obtain f and Y^I from (3.27), one has to solve the following quadratic equations for functions y^I

$$D_{IJK} y^I y^J = -2K_I K^0 + D_{IJK} K^J K^K \quad (3.32)$$

from which f and Y^I are obtained as

$$f = \frac{2^{2/3}K^0}{\mathcal{Q}}, \quad Y^I = \frac{2^{1/3}y^I}{\sqrt{\mathcal{Q}}}, \quad \mathcal{Q} = \left(\frac{1}{3} D_{IJK} y^I y^J y^K \right)^{2/3}. \quad (3.33)$$

The examples we will consider in section 4 below fall into a simple subclass of solutions where all the K^I and all the K_I are proportional to each other, $K^I = p^I \bar{K}$, $K_I = p_I \underline{K}$, with $p_I = D_{IJK} p^J p^K$. This ansatz leads in particular to solutions where the vector multiplet scalars Y^I are constant:

$$Y^I = \left(\frac{6}{p^3} \right)^{\frac{1}{3}} p^I, \quad (3.34)$$

where we have defined

$$p^3 \equiv D_{IJK} p^I p^J p^K. \quad (3.35)$$

When the axidilaton is constant, this gives an attractor solution where the asymptotic moduli are fixed at their attractor values. Although not much is known about the attractor mechanism in the presence of hypermultiplets, it seems likely to us that such solutions are still attractors in the presence of the axidilaton.

Now let's discuss the differential equations which the various ingredients in the solution (3.25)–(3.30) must satisfy. The main difference with solutions with a translational Killing vector (such as those with a Gibbons-Hawking base) is that the functions K^0, K^I, K_I, K_0 are not harmonic with respect to the 3D metric (3.22) but instead satisfy

$$\Delta_{s_2} K^0 \equiv d \star_3 (dK^0 + s_2 (K^0)^2 dy_2) = 0 \quad (3.36)$$

$$\Delta_{s_2} K^I \equiv d \star_3 (dK^I + s_2 K^0 K^I dy_2) = 0 \quad (3.37)$$

$$\Delta_{s_2} K_I \equiv d \star_3 \left(dK_I + s_2 (K^0 K_I + \frac{1}{2} D_{IJK} y^J y^K) dy_2 \right) = 0 \quad (3.38)$$

$$\Delta_{s_2} K_0 \equiv d \star_3 \left(dK_0 + \frac{s_2}{2} (K^I K_I - K^0 K_0) dy_2 + \frac{s_2}{2} \star_3 (\omega \wedge dy_2) \right) = 0 \quad (3.39)$$

while the one-forms χ, ω satisfy

$$\star_3 d\chi = dK^0 + s_2 (K^0)^2 dy_2 \quad (3.40)$$

$$\star_3 d\omega = \langle dK, K \rangle - s_2 L dy_2. \quad (3.41)$$

where, in the second line, we have viewed $K = (K^0, K^I, K_I, K_0)$ as a vector in a space equipped with a symplectic inner product

$$\langle A, B \rangle = -A^0 B_0 + A^I B_I - A_I B^I + A_0 B^0 \quad (3.42)$$

In these expressions \star_3 is the 3D Hodge star taken with respect to the orientation $(\text{Re}w^1, \text{Im}w^1, y_2)$. The equations (3.36), (3.40) follow from the Toda-like equation (3.18) and the definitions (3.19), (3.20). Note that we have introduced for later convenience the shorthand notation Δ_s to represent the differential operators acting on the K functions; one

should keep in mind that the action of Δ_s depends on which component of the symplectic vector K it acts. We can then abbreviate (3.36)–(3.39) to

$$\Delta_{s_2} K = 0. \quad (3.43)$$

Interestingly, the equations (3.37)–(3.39) are invariant under an n_V -parameter family of solution generating transformations:

$$K^I \rightarrow K^I + k^I K^0 \quad (3.44)$$

$$K_I \rightarrow K_I + D_{IJK} k^J K^K + \frac{1}{2} D_{IJK} k^J k^K K^0 \quad (3.45)$$

$$K_0 \rightarrow K_0 + k^I K_I + \frac{D_{IJK}}{6} (k^I K^J K^K + 3k^I k^J k^K K^0) \quad (3.46)$$

with k^I arbitrary real constants. We note that the quantity L defined in (3.29) is invariant under these transformations, so that from (3.41) one easily verifies that ω is also invariant. This symmetry is a generalization of the ‘spectral flow’ symmetry in solutions with a translational Killing vector [24–26].

Now let’s discuss the integrability condition coming from (3.41). Applying $d\star_3$ on both sides we find the condition

$$0 = dd\omega = \langle \Delta_{s_2} K, K \rangle \quad (3.47)$$

which is of course automatically satisfied if (3.43) holds with all components of K being smooth, i.e. without δ -function terms on the right hand side. More interesting is the case is where one allows such δ -functions sources and replaces (3.43) with

$$\Delta_{s_2} K = \sum_i \Gamma_i \delta^3(x - x_i) \text{vol}_3 \quad (3.48)$$

Such singularities correspond in the M-theory language to turning on, in addition to possible M2-branes in the noncompact directions which source the axidilaton, other M-brane and momentum/KK monopole charges Γ at the positions x_i on the 3D base which source the vector multiplets:³ Γ^0 corresponds to a KK monopole, Γ^I to an M5 on $\mathcal{C}^I \times S_{\theta^2}^1$, Γ_I to an M2 on \mathcal{C}_I and Γ_0 to momentum along $S_{\theta^2}^1$. Note that when s_2 is zero these point charges are the only ones present, while in the case of $s_2 \neq 0$ additional smeared charge may appear. It is very remarkable however, that even when $s_2 \neq 0$ the well known stability equations [27, 28] remain functionally the same. These equations follow from the the integrability condition (3.47) which, in the presence of delta-function sources (3.48), imposes nontrivial constraints on the allowed charges and positions of the branes in the 3D submanifold. Indeed, for each center we must impose

$$\langle \Gamma_i, K(x_i) \rangle = 0. \quad (3.49)$$

When θ^2 is periodic, the solutions with rotational Killing vector can be dimensionally reduced to 4D along the θ^2 direction, yielding an as yet unexplored and potentially interesting class of multicentered solutions carrying various brane charges. One open question

³The \mathcal{C}_I represent a basis of two-cycles on the Calabi-Yau, with \mathcal{C}^I the dual basis of four-cycles with respect to the intersection product, and $S_{\theta^2}^1$ is the θ^2 circle.

regarding such 4D solutions is whether they are still supersymmetric. Since for a rotational Killing vector the 5D Killing spinor depends on θ^2 (this follows from the θ^2 dependence of Φ^\pm which are bilinears in the Killing spinor), it is not clear whether the reduced 4D solution will preserve supersymmetry in general. Furthermore, upon dimensional reduction one obtains a 4D metric which is a timelike fibration over the 3D metric ds_3^2 given in (3.22). Even for constant axidilaton, we know from (3.24) that this 3D metric is not flat in general. The reduced 4D metric is then not obviously of the form introduced by Tod [29, 30] which was shown to govern general supersymmetric solutions with vector multiplets [31]. We feel that this interesting issue deserves further investigation.

The structure of the solutions simplifies considerably when the Killing vector is translational, which can be obtained as the limit $s_2 \rightarrow 0$ of the expressions above. The 4D metric (3.21) reduces to

$$ds_4^2 = K^0 ds_3^2 + \frac{1}{K^0} (d\theta^2 + \chi)^2 \quad (3.50)$$

$$ds_3^2 = dy_2^2 + \tilde{\tau}_2 e^{\tilde{h}_1} dw^1 d\bar{w}^1 \quad (3.51)$$

where, in view of (3.36), K^0 is now a harmonic function of the 3D geometry, and $\star_3 d\chi = dK^0$. These metrics are therefore generalizations of the Gibbons-Hawking metrics [32], where the 3D base manifold is generically curved due to the factor $\tilde{\tau}_2$ in (3.51), see (3.24). The form of the solution to the remaining equations determining the full 5D solution also simplifies significantly in the case of a translational Killing vector. The solution can still be written in the form (3.25)–(3.30), but now all functions are harmonic in the (generically curved) 3D metric (3.84), and the equation determining ω also simplifies. Summarized, the equations (3.43), (3.41) are now replaced by

$$\nabla_3^2 K = 0 \quad (3.52)$$

$$\star_3 d\omega = \langle K, dK \rangle. \quad (3.53)$$

When the axidilaton is constant, the 3D metric (3.51) becomes flat and the metric on the base is a Gibbons-Hawking hyperkähler metric [32]. These solutions arise as 5D uplifts [23, 25, 33–35] of the 4D N=2 vector multiplet solutions of [36] describing type IIA multicentered configurations of branes wrapped on the Calabi-Yau cycles. The constraints (3.49) are the well-known stability equations governing the existence of supersymmetric bound states [27]. A subset of these uplifted solutions are the 5D smooth bubbling geometries of [28] containing topologically nontrivial cycles.

Since in the case of a translational isometry this analysis has led to a wealth of insights in the BPS spectrum of string/M theory and phenomena such as wall-crossing [37], it would be of great interest to get a handle on the constraints (3.49) on multicenter solutions with a rotational isometry. These may also play a role in constructing horizonless microstate geometries carrying the same charges as black holes or black rings (see e.g. [38] for a review and further references). In particular, the geometries in the black hole deconstruction proposal of [8] are multi-centered solutions of (3.36)–(3.41) with a rotational isometry and a nontrivial axidilaton from an M2 brane in the bulk. We will come back to this proposal in section 4.2.5.

Since the hypermultiplets enter in (2.11) only implicitly through their impact on the 4D base metric, one would expect that the results we derived in this section for axidilaton solutions can be generalized to more general solutions involving hypermultiplets invariant under an isometry.

We end this section by comparing our results to those obtained in [3], which also considered supersymmetric solutions with an extra isometry under which τ is invariant. In that work however, the Killing vector in question was tacitly assumed to be translational, corresponding to the $s_2 = 0$ solutions (3.50)–(3.53) in the present discussion, see in particular eqs. (4.83) in [3]. In uplifts of 4D multicentered solutions, the Killing vector which generates translations on the M-theory circle is of the translational type. The restricted class of solutions considered in [3] is suited to describe, for example, backreacted M2 branes which are either localized or smeared on the M-theory circle. Our generalization (3.18), (3.25)–(3.41) on the other hand is needed to describe configurations of M2 branes which are localized on the M-theory circle but which do possess a rotational Killing vector leaving τ invariant. We will discuss an explicit example in section 4.2.4. This class also contains the brane configurations in the black hole deconstruction proposal of [8], on which we will comment in section 4.2.5.

3.2.3 Separated Toda solutions and enhanced symmetries

In the case of a rotational Killing vector, $s_2 \neq 0$, the geometry of the base is governed by the generalized Toda equation (3.18). Even when the axidilaton is constant, only few explicit solutions to this equation are known. In this section we will concentrate on a special class of solutions to (3.18), where the Toda potential is of the separated form

$$e^{s_2 \Psi(w^1, \bar{w}^1, y_2)} = g(y_2) e^{-2\Phi(w^1, \bar{w}^1)}. \quad (3.54)$$

with $g(y_2)$ a positive real function. For constant τ , this ansatz leads to the Liouville equation for Φ [11] and hence to simple explicit solutions to (3.18). We will find that also when τ is turned on the separated ansatz leads to a more tractable subclass which includes metrics with additional symmetries. We will now derive these symmetry properties and give some explicit solutions to the equations (3.36)–(3.41) which lead to highly symmetric 5D solutions.

With $e^{s_2 \Psi}$ of the factorized form (3.54), the equation (3.18) implies a deformed Liouville equation for Φ :

$$\begin{aligned} 4\Phi_{w^1 \bar{w}^1} - \kappa^2 \tilde{\tau}_2 e^{\tilde{h}_1} e^{-2\Phi} &= 0 \\ g'' &= 2\kappa^2 \end{aligned} \quad (3.55)$$

where κ^2 is a real constant.⁴ Recall that $\tilde{\tau}$ and \tilde{h} are holomorphic functions of w^1 . The base metric is of the form (3.21) with

$$ds_4^2 = K^0(dy_2^2 + gds_2^2) + \frac{1}{K^0} (d\theta^2 + \chi)^2 \quad (3.56)$$

$$K^0 = \frac{g'}{s_2 g}; \quad \chi = \frac{4}{s_2} \text{Im}(\Phi_{w^1} dw^1) \quad (3.57)$$

$$ds_2^2 = \tilde{\tau}_2 e^{\tilde{h}_1} e^{-2\Phi} dw^1 d\bar{w}^1 \quad (3.58)$$

⁴By a shift of Φ we could set κ^2 to either -1 , 1 or 0 , but we will find it convenient to keep κ around.

Let's first review the case when the axidilaton is constant [11]. Then the Liouville equation (3.55) implies that the 2D metric (3.58) has constant curvature, $R^{(2)} = 2\kappa^2$. Since 2D spaces of constant curvature are locally isomorphic to either the hyperbolic plane, the two-sphere or the plane depending on the sign of κ^2 , the solution has (locally) an additional three dimensional algebra of Killing vectors, namely $so(3)$, $sl(2, \mathbb{R})$ or the euclidean algebra $e(2)$ respectively. In fact, when $\kappa^2 = 0$, one can check that the base is completely flat and has local symmetry $e(4)$.

A natural question is thus whether it is possible to preserve this additional symmetry when the axidilaton becomes dynamic. Again, for this to be the case it is necessary that the 2D metric (3.58) has constant curvature. Using the equation (3.55) one finds that the 2D curvature of a generic solution is

$$R^{(2)} = 2\kappa^2 + \frac{|\tilde{\tau}'|^2 e^{2\Phi}}{\tilde{\tau}_2^3 e^{\tilde{h}_1}} \quad (3.59)$$

To have constant curvature the second term should be a constant, which has to be determined by demanding compatibility with (3.55). One finds that a constant curvature solution is possible only for negative κ^2 and is given by

$$\Phi = \frac{1}{2} \ln \left(-\frac{2\kappa^2 \tilde{\tau}_2^3 e^{\tilde{h}_1}}{3|\tilde{\tau}'|^2} \right), \quad \text{for } \kappa^2 < 0. \quad (3.60)$$

for which the 2D metric (3.58) has curvature $R^{(2)} = 4\kappa^2/3$. This solution to the deformed Liouville equation (3.55) was considered in [9] (and in a different context in [39]) and, as we will discuss in section 4.2.4, gives rise to 5D solutions which are (locally) Gödel $\times S^2$. It is furthermore the unique solution to the equation (3.55) on a compact manifold without boundary [40]. However, as we will see explicitly in section 4.2.5 and appendix C, (3.60) is not the unique solution to (3.55) in the presence of a boundary, which will be the situation of interest to us. For the other solutions to (3.55) the 2D metric ds_2^2 has non-constant curvature (3.59) and generically doesn't possess additional Killing vectors.

Let us now discuss how to construct full 5D solutions on a base determined by a separated Toda solution. First we should point out that the separated ansatz contains 'ambipolar' base metrics whose signature changes from mostly plus in one region to mostly minus in another. When this happens the 4D base is singular, as the metric eigenvalues pass through zero, but it is often possible to turn on vector multiplet fluxes so as to give completely regular 5D metrics [28]. From the form of the metric (3.56) we see that the metric is ambipolar if and only if K^0 changes sign. It's easy to see that this can happen only when $\kappa^2 \neq 0$. In this case by shifting y_2 with a constant we can assume that the function g is

$$g = \kappa^2 y_2^2 + 4a^2 \quad \text{when } \kappa^2 \neq 0 \quad (3.61)$$

with a^2 a real number. The range of y_2 must be chosen such that g is positive. Hence we see the base is ambipolar when $\kappa^2 \neq 0$ and $a^2 > 0$. Summarized, we have

$$\kappa^2 = 0 \text{ or } a^2 < 0 < \kappa^2 \Leftrightarrow \text{positive signature base}, \quad (3.62)$$

$$\kappa^2 \neq 0, a^2 > 0 \Leftrightarrow \text{ambipolar base}. \quad (3.63)$$

Let us also note that the Kähler potential can be found explicitly by making a Legendre transform. For g given by (3.61) the result is

$$\mathcal{K} = \frac{4a}{\kappa s_2} \left(\sqrt{\frac{e^{s_2 x^2 + 2\Phi}}{4a^2} - 1} - \arctan \sqrt{\frac{e^{s_2 x^2 + 2\Phi}}{4a^2} - 1} \right). \quad (3.64)$$

One checks that the Monge-Ampère equation (3.9) is satisfied provided that (3.55) holds.

The next step in finding complete 5D solutions is to solve equations (3.37)–(3.41). When the base has positive signature, we can always trivially extend it to a five-dimensional static solution with trivial vector multiplets by taking $K^I = K_I = K_0 = \omega = 0$, so that $ds_5^2 = -dt^2 + ds_4^2$.

More interesting 5D solutions are obtained from ambipolar 4D bases. As remarked above, in this case we have to turn on vector multiplets if we want to have a chance of obtaining a regular 5D solution. It turns out that, from every factorized solution (3.54) with ambipolar base it is possible to build a 5D solution (3.37)–(3.41) where either the timelike Killing vector ∂_t (for $\kappa^2 > 0$) or the spacelike Killing vector ∂_{θ^2} (for $\kappa^2 < 0$) are part of an extended 3-dimensional symmetry algebra. The construction goes as follows. Since $K^0 = \frac{g'}{s_2 g}$ depends only on y_2 , it is natural to look for solutions where K^I, K_I and K_0 also depend only on y_2 . Restricting attention to solutions of the form (3.34) which have constant vector multiplet scalars, and making suitable choices for the integration constants for the resulting equations,⁵ one finds the following solution to (3.37)–(3.39):

$$\begin{aligned} K^I &= \frac{2ap^I}{g}, & K_I &= -\frac{s_2^2 p_I}{8\kappa^2} K^0 \\ K_0 &= -\frac{s_2^2 ap^3}{12\kappa^2} \left(\frac{1}{g} - \frac{b}{2a^2} \right), & L &= \frac{p^3}{6ag^2} (4a^2 + b\kappa^2 y_2^2) \\ \omega &= -\frac{s_2^2 bp^3}{24a\kappa^2} \chi + 2\lambda, & d\lambda &= 0 \end{aligned} \quad (3.65)$$

The solution depends on the free parameters p^I (with p^3 as defined in (3.35)) and b and on a closed one-form λ . When λ is exact, it can be absorbed in a redefinition of the time coordinate. A non-exact one-form λ will be needed to ensure that ω has no Dirac string singularities and the integrability condition (3.47) is satisfied, which also has the effect of removing closed timelike curves [22, 25]. We will discuss the required form of λ in more detail in section 3.3.4. The resulting 5D solution is

$$ds_5^2 = \left(\frac{p^3}{6} \right)^{2/3} \left[\frac{dy_2^2}{g} - \frac{g}{4a^2 \kappa^2} \alpha^2 + \frac{1}{\kappa^2} \left(\alpha + \frac{s_2}{2} (\chi + d\theta^2) \right)^2 + \tilde{\tau}_2 e^{\tilde{h}_1} e^{-2\Phi} dw^1 d\bar{w}^1 \right] \quad (3.66)$$

$$F^I = \frac{p^I}{2a} dy_2 \wedge \alpha, \quad Y^I = \left(\frac{6}{p^3} \right)^{\frac{1}{3}} p^I \quad (3.67)$$

$$\alpha = -\frac{bs_2}{2} d\theta^2 - \frac{24a\kappa^2}{s_2 p^3} (dt + \lambda), \quad d\lambda = 0 \quad (3.68)$$

with g given in (3.61) with $\kappa^2 \neq 0$, $a^2 > 0$ and Φ a solution to (3.55). Recall that on an ambipolar base a is real, ensuring that the complete solution is indeed manifestly real. The

⁵See [22], section 6.1 for a discussion of the most general solution where K depends only on y_2 .

first two terms in the metric (3.66) constitute a 2D metric of curvature $-2\kappa^2$, so that for $\kappa^2 < 0$, the solution contains an AdS_2 subspace fibered over a 3D base, while for $\kappa^2 > 0$ it contains a (fibered) round S^2 . The vector multiplet flux is supported on AdS_2 or S^2 respectively. The latter case is a ‘bubbled solution’ in the spirit of [28] and will play an important role later on. Hence we have demonstrated the existence of bubbled solutions even in the presence of hypermultiplets.

3.3 Solutions with toric Kähler base

The discussion in the last subsection is as far as we managed to go for generic solutions with a single space-like Killing vector. Here we will see how demanding the presence of a second Killing vector, commuting with the first, constrains the solutions much more. The base becomes a (generalized) toric Kähler manifold, and furthermore the possible profiles for τ are completely fixed by the symmetry up to some free constants and a few discrete choices. After working out those observations in the first subsection we illustrate them in the simple case with constant axidilaton, when the geometry is actually hyperkähler. We then move on to dynamic axidilaton configurations, showing how the profiles we derived from symmetry have a physical interpretation as the presence of M2/exotic brane sources. Finally we use the separated solutions of the Toda equation to provide complete 5D solutions in the case with 2 Killing vectors.

3.3.1 Toric Kähler manifolds and axidilaton profiles

Having analyzed the structure of solutions with a single compatible (in the sense of section 3.1) Killing vector, we can go one step further and impose that the 4D base has two commuting Killing vectors $k^{(1)}, k^{(2)}$. We choose complex coordinates

$$w^i = x^i + i\theta^i \quad (3.69)$$

which are adapted to the isometries, $k^{(i)} = \frac{\partial}{\partial \theta^i}$, and locally pick a Kähler potential which is independent of both θ^i :

$$\mathcal{K} = \mathcal{K}(x^i). \quad (3.70)$$

We then have, locally, from (3.10)

$$i_{k^{(i)}} \Phi_3 = -dy_i \quad (3.71)$$

with $y_i \equiv \mathcal{K}_{x^i}$. We will make here the extra technical assumption that the y_i extend to globally well-defined functions, so that our Killing vectors are Hamiltonian and y_i are the moment maps corresponding to k^i . The base is then a toric⁶ Kähler manifold. The restriction to toric 4D bases includes, as we will see below, a number of physically interesting

⁶To be precise we should speak of a generalized toric manifold. The most conservative mathematical definition requires a $2n$ -dimensional compact symplectic manifold with a Hamiltonian n -torus action, so that the image of the moment map is a convex polygon by the Atiyah-Guillemin-Sternberg theorem. This definition can be relaxed however to the non-compact setting, see e.g. [41], where in return it is demanded that the moment map be proper onto its convex image, to ensure some polytopical form for that image. Here we will be a bit more loose in our nomenclature and for us a toric manifold will simply be any symplectic $2n$ dimensional manifold with a Hamiltonian \mathbb{T}^n action.

Class	$\tau(w^1)$	$h - s_i w^i$	$e^{\mu(x^1)} \equiv \tau_2 e^{h_1 - s_i x^i}$
II	i	0	1
IP	$i(1 - pw^1)$	0	$1 - px_1$
IH	ie^{-irw^1}	irw^1	$\cos rx_1$
IE	$i \tanh(1 - qw^1)$	$2 \ln \cosh(1 - qw^1)$	$\frac{1}{2} \sinh 2(1 - qx^1)$

Table 3. The profiles for τ, h and the function μ for solutions with toric Kähler base.

situations and has the advantage of simplifying the BPS equations. In particular, the configurations we are most interested in will be governed by solutions of an ordinary nonlinear differential equation (see (3.125) below). In addition, noncompact toric Kähler manifolds are a subject of recent interest in the mathematics community, see [41] and references therein.

Let's first discuss how the toric symmetry restricts the factor $\tau_2 e^{h_1}$ in the Monge-Ampère equation (2.32). Repeating the analysis of section 3.1 for a second Killing vector, one finds that the function $\tilde{\tau}$ in table 1 must be a linear function of w^2 . Making a linear redefinition of the complex coordinates, we can arrange that τ depends only on w^1 . In this coordinate frame $k^{(2)} = \frac{\partial}{\partial \theta^2}$ leaves τ invariant, while under the action of $k^{(1)} = \frac{\partial}{\partial \theta^1}$, τ is either invariant or transforms by a parabolic, hyperbolic or elliptic U-duality transformation. We will call these four cases *II*, *IP*, *IH* and *IE* respectively. The expressions for τ and h then reduce to (3.4) and (3.6) with $\tilde{\tau}(w^2) = \tau_0$ and $\tilde{h}(w^2) = s_2 w^2 + \ln c$, with τ_0, c constants, so that they are fully determined by the symmetries. In what follows we will set $\tau_0 = i$ and $c = 1, p, q, r$ in classes *II*, *IP*, *IH* and *IE* respectively. This choice will have the advantage that both τ and h remain well-defined in the limit $p, q, r \rightarrow 0$, a fact which will be useful later. We display the resulting expressions for τ and h with this choice of integration constants in table 3. It will also be useful to single out the following combination, which depends only on x^1 :

$$e^{\mu(x^1)} \equiv \tau_2 e^{h_1 - s_i x^i} = \begin{cases} 1 & \text{case II} \\ \frac{c\tau_2}{\sqrt{\tau' \tilde{\tau}'}} & \text{cases IP, IH, IE} \end{cases} \quad (3.72)$$

Note that μ solves a real Liouville equation:

$$\mu'' + c^2 e^{-2\mu} = 0. \quad (3.73)$$

Explicit expressions for μ are also given in table 3.

The meaning of the constants s_1, s_2 is as follows. When both s_1 and s_2 are zero, both Killing vectors are translational. Note that when both s_1 and s_2 are nonzero, there is one linear combination of the Killing vectors (namely $s_2 k^{(1)} - s_1 k^{(2)}$) which is translational while another combination is rotational. Therefore, from the moment that either s_1 or s_2 is nonzero we have one rotational and one translational Killing vector.

We observe that in all cases $\tau_2 e^{h_1}$ is independent of both θ^i , which is consistent with the property (3.70) and the equation (2.32) for the Kähler potential. In particular, in the toric case the equation (2.32) becomes a real Monge-Ampère equation:

$$\mathcal{K}_{x^1 x^1} \mathcal{K}_{x^2 x^2} - (\mathcal{K}_{x^1 x^2})^2 = e^{\mu + s_1 x^1 + s_2 x^2} \quad (3.74)$$

and the 4D base metric can be written as

$$ds_4^2 = \mathcal{K}_{ij} dx^i dx^j + \mathcal{K}_{ij} d\theta^i d\theta^j. \quad (3.75)$$

The toric Kähler geometry can also be described in terms of symplectic coordinates y_i, θ^i in terms of which the Kähler form takes the canonical (Darboux) form:

$$\Phi_3 = dy_i \wedge d\theta^i \quad (3.76)$$

i.e. the y_i play the role of canonical momenta conjugate to the torus coordinates θ^i . The y_i are the moment maps of the Killing vectors k^i which, as we argued in (3.71), are simply the derivatives of the Kähler potential:

$$y_i = \mathcal{K}_{x^i}. \quad (3.77)$$

In symplectic coordinates the geometry is encoded in a symplectic potential \mathcal{S} which is related to the Kähler potential by a Legendre transform with respect to x^1 and x^2 :

$$\mathcal{S}(y_i) = x^i y_i - \mathcal{K} \quad (3.78)$$

$$ds_4^2 = \mathcal{S}^{ij} dy_i dy_j + \mathcal{S}_{ij}^{-1} d\theta^i d\theta^j, \quad \mathcal{S}^{ij} \equiv \frac{\partial^2 \mathcal{S}}{\partial y^i \partial y^j} \quad (3.79)$$

The coordinates y_i trace out a convex region in \mathbb{R}^2 called the *moment polytope*. It is determined by the requirement that

$$\det \mathcal{S}_{ij}^{-1} \geq 0. \quad (3.80)$$

The edges of the moment polytope form the locus where the torus degenerates. The symplectic potential also satisfies a Monge-Ampère-type equation, namely

$$\mathcal{S}_{y_1 y_1} \mathcal{S}_{y_2 y_2} - (\mathcal{S}_{y_1 y_2})^2 = e^{-\mu(\mathcal{S}_{y_1}, \mathcal{S}_{y_2}) - s_1 \mathcal{S}_{y_1} - s_2 \mathcal{S}_{y_2}} \quad (3.81)$$

There is a third description of the toric Kähler base which is the most useful for finding explicit solutions. Since the axidilaton is left invariant by one of the Killing vectors, which in our conventions is $k^{(2)} = \partial_{\theta^2}$, we can describe the geometry in terms of a Toda potential as we discussed in section 3.2. That discussion goes through unchanged in the toric case, the only difference being that all quantities are now independent of θ^1 . In particular, the Toda potential $\Psi(x^1, y_2)$ now satisfies a Toda-like differential equation in two real variables

$$\Psi_{x^1 x^1} + \frac{e^{\mu + s_1 x^1}}{s_2} (e^{s_2 \Psi})_{y_2 y_2} = 0 \quad (3.82)$$

The base metric (3.21) simplifies to

$$\begin{aligned} ds_4^2 &= K^0 ds_3^2 + \frac{1}{K^0} (d\theta^2 + \chi)^2 \\ K^0 &= \Psi_{y^2}, \quad \chi = -\Psi_{x^1} d\theta^1 \\ ds_3^2 &= dy_2^2 + e^{\mu+s_1 x^1+s_2 \Psi} ((dx^1)^2 + (d\theta^1)^2) \end{aligned} \quad (3.83)$$

Given a solution to (3.82) leading to a toric 4D base, one can look for full 5D solutions preserving the toric isometries of the base by solving the equations (3.36)–(3.39) for functions K^I, K_I, K_0 which are independent of θ^1 .

3.3.2 Toric hyperkähler from Gibbons-Hawking

To illustrate the equations and solutions with toric symmetry we first consider a class of examples with constant axidilaton, where the 4D hyperkähler base is a Gibbons-Hawking manifold. Such solutions have a translational isometry, and we will look here at the subclass which has an extra rotational symmetry, so that the base is toric hyperkähler with both a translational and a rotational Killing vector. This is the case for example for a multi-Taub-NUT solution where all the centers lie on an axis, and more generally any Gibbons-Hawking metric constructed from an axially symmetric harmonic function is a toric hyperkähler metric. Although this is a known result in the mathematics literature [41], we will rederive it here in a way that is completely explicit. In such solutions, the base can be written both in the Gibbons-Hawking form, where the base is fibered along the translational direction over a flat 3D base, and the Toda form (3.83), by taking the fiber to be the rotational S^1 . We derive here how these are related, and construct an (implicit) solution to the Toda equation (3.82) for every axially symmetric harmonic function. We then extend this correspondence to full 5D solutions, yielding an explicit non-trivial solution for the quantities K, ω introduced in section 3.2.2 in terms of axisymmetric harmonic functions.

Our starting point is the well-known Gibbons-Hawking hyperkähler metric, with an additional axial symmetry in the 3D flat base. In coordinates where ∂_{θ^1} generates the translational symmetry and ∂_{θ^2} generates the axial one the metric takes the form

$$ds_4^2 = H^0 (dr^2 + r^2 (d\theta^2)^2 + dz^2) + \frac{1}{H^0} (d\theta^1 + \tilde{H}^0 d\theta^2)^2 \quad (3.84)$$

where H^0 is an axially symmetric harmonic function (depending only on r and z). The function \tilde{H}^0 is defined as follows. For any axially symmetric harmonic function H one can define a conjugate function \tilde{H} (see e.g. [42]) satisfying $dH = \star_3 d(\tilde{H} d\theta^2)$. In other words,

$$r \partial_r H = -\partial_z \tilde{H} \quad (3.85)$$

$$r \partial_z H = \partial_r \tilde{H}. \quad (3.86)$$

Integrability of these equations imposes that H, \tilde{H} satisfy the second order equations

$$(\partial_r^2 + r^{-1} \partial_r + \partial_z^2) H = 0 \quad (3.87)$$

$$(\partial_r^2 - r^{-1} \partial_r + \partial_z^2) \tilde{H} = 0. \quad (3.88)$$

It is not hard to check that $k^{(1)} = \partial_{\theta^1}$ is indeed a translational or triholomorphic Killing vector, which is equivalent to the associated one-form having self-dual curvature:

$$dk^{(1)} = \star_4 dk \quad k_a^{(1)} = g_{a\theta^1}. \quad (3.89)$$

This means this solution falls into the classification of section 3.1 with

$$s_1 = 0. \quad (3.90)$$

The second Killing vector $k^{(2)} = \partial_{\theta^2}$ is still holomorphic with respect to one of the complex structures, but no longer triholomorphic, making it rotational: $s_2 \neq 0$. The analysis of section 3.2.1 then implies there should also exist a Toda form (3.83) for the metric, which in this case is

$$ds_4^2 = \Psi_{y_2} ((dy_2)^2 + e^{s_2\Psi} ((dx^1)^2 + (d\theta^1)^2) + \frac{1}{\Psi_{y_2}} (d\theta^2 - \Psi_{x^1} d\theta^1)^2) \quad (3.91)$$

As the two metrics (3.91) and (3.84) describe the exact same geometry we should be able to related them by a coordinate transformation. This might sound trivial, but it implies a rather intricate relation between solutions of the non-linear Toda equation and the simple linear Laplace equation in flat space. We start by equating the $d\theta^i d\theta^j$ terms in (3.84) and (3.91), this gives the algebraic relations

$$e^{s_2\Psi} = r^2 \quad (3.92)$$

$$\Psi_{x^1} = -\frac{\tilde{H}^0}{(H^0)^2 r^2 + (\tilde{H}^0)^2} \quad (3.93)$$

$$\Psi_{y_2} = \frac{H^0}{(H^0)^2 r^2 + (\tilde{H}^0)^2}. \quad (3.94)$$

Note that $e^{s_2\Psi/2}$ has the interpretation of the distance to the axis of symmetry in the flat metric. Hence, to find the Toda potential, we simply have to solve for the radial distance r in terms of the variables x^1, y_2 . Equating the remaining terms in (3.84) and (3.91) then leads to the following relations between the coordinates:

$$\begin{aligned} dx^1 &= -\frac{\tilde{H}^0}{r} dr + H^0 dz \\ dy_2 &= H^0 r dr + \tilde{H}^0 dz. \end{aligned} \quad (3.95)$$

Compatibility of these relations and (3.92) with with equations (3.93), (3.94) fixes the rotational charge s_2 to be

$$s_2 = 2. \quad (3.96)$$

A nice consistency check is to observe that (3.86) are exactly the conditions that the relations (3.95) can be integrated, i.e. they imply that $ddx^1 = ddy_2 = 0$ so that x^1, y_2 are well-defined functions. Another interesting observation is that the functions x^1, y_2 form a new pair of conjugate harmonic functions, since

$$r\partial_r x^1 = -\partial_z y_2 \quad (3.97)$$

$$r\partial_z x^1 = \partial_r y_2 \quad (3.98)$$

The couple x^1, y_2 is the ‘primitive’ pair of conjugate harmonic functions constructed from the pair H^0, \tilde{H}^0 [42]. These relations let us also express the conditions (3.86) for a pair of functions (H, \tilde{H}) to be a harmonic pair, in the new coordinates:

$$\partial_{x^1} H = \partial_{y_2} \tilde{H} \quad (3.99)$$

$$\partial_{x^1} \tilde{H} = -e^{s_2 \Psi} \partial_{y_2} H \quad (3.100)$$

Using the relations (3.94) one can check that the above equations are equivalent to Ψ satisfying the Toda equation (3.18). Furthermore we have the relations

$$\begin{aligned} y_1 &= z \\ x^2 &= \ln r. \end{aligned} \quad (3.101)$$

Let us also comment on how to find the Kähler and symplectic potentials. As discussed in section 3.2.1, these can be obtained from either H^0 or Ψ by integrating and making a Legendre transformation. In practice however, it is simpler to first find the relation between the Kähler coordinates x^1, x^2 and the symplectic coordinates y_1, y_2 through the relations (3.95), (3.101) and then integrate the equations

$$\mathcal{K}_{x^1} = y_1(x^1, x^2) \quad \mathcal{K}_{x^2} = y_2(x^1, x^2) \quad (3.102)$$

$$\mathcal{S}_{y_1} = x^1(y_1, y_2) \quad \mathcal{S}_{y_2} = x^2(y_1, y_2) \quad (3.103)$$

We can make these considerations more concrete for multi-centered Gibbons-Hawking bases, where all centers all lie on the z -axis:

$$H^0 = h^0 + \sum_i \frac{q_i^0}{\sqrt{r^2 + (z - z_i)^2}} \quad (3.104)$$

$$\tilde{H}^0 = \sum_i \frac{q_i^0(z - z_i)}{\sqrt{r^2 + (z - z_i)^2}} \quad (3.105)$$

$$x^1 = hz + \frac{1}{2} \sum_i q_i^0 \ln \frac{\sqrt{r^2 + (z - z_i)^2} + (z - z_i)}{\sqrt{r^2 + (z - z_i)^2} - (z - z_i)} \quad (3.106)$$

$$x^2 = \Psi = \ln r \quad (3.107)$$

$$y_1 = z \quad (3.108)$$

$$y_2 = \frac{hr^2}{2} + \sum_i q_i^0 \sqrt{r^2 + (z - z_i)^2} \quad (3.109)$$

In principle expressions for the Toda, Kähler and symplectic potentials can be found by inverting the relations above. But it looks hard, if not impossible, to explicitly relate the coordinates x^i and y_i , in the case of more than two centers. Hence the implicit description above is as far as we can go in general. For up to two centers with equal or opposite charges, an explicit description is possible and we will review these solutions in section 4.

Now that we have ‘solved’ for the base in the Toda form we can go on and extend these relations to all quantities appearing in the full 5D supergravity solution discussed

in section 3.2.2. In ‘Gibbons-Hawking form’ they are determined in terms of a number of harmonic functions⁷ $H = (H^0, H^I, H_I, H_0)$ as

$$ds_5^2 = -f^2(dt + \xi)^2 + f^{-1}ds_4^2 \quad (3.110)$$

$$\Theta^I = \left(-2H^0 \star_3^{GH} d\left(\frac{H^I}{H^0}\right) \right)^- \quad (3.111)$$

$$f^{-1}Y_I = -2H_I + D_{IJK}\frac{H^J H^K}{H^0}, \quad D_{IJK}Y^I Y^J Y^K = 6 \quad (3.112)$$

$$\xi = \frac{\omega^{GH}}{2} + \frac{L^{GH}}{2(H^0)^2}(d\theta^1 + \tilde{H}^0 d\theta^1) \quad (3.113)$$

$$\star_3^{GH} d\omega^{GH} = \langle dH, H \rangle \quad (3.114)$$

$$L^{GH} = H_0(H^0)^2 + \frac{1}{3}D_{IJK}H^I H^J H^K - H^I H_I H^0 \quad (3.115)$$

$$F^I = -d(fY^I(dt + \xi)) + \Theta^I \quad (3.116)$$

When all harmonic functions are of the Coulomb form (3.104) with delta-function sources on the z -axis, the integrability condition (3.49) for (3.114) leads to the following constraint on the position of the centers:

$$\sum_{j \neq i} \frac{\langle q_i, q_j \rangle}{|z_i - z_j|} = \langle h, q_i \rangle \quad (3.117)$$

for each of the centers labeled by i .

One can now find the expressions in the ‘Toda form’ by comparing to the solution (3.25)–(3.30) for non-zero s_2 . Equating for example the two expressions for Θ^I one finds the relations

$$\partial_r \left(\frac{K^I}{K^0} \right) = \tilde{H}^0 \partial_r \left(\frac{H^I}{H^0} \right) - r H^0 \partial_z \left(\frac{H^I}{H^0} \right) \quad (3.118)$$

$$\partial_z \left(\frac{K^I}{K^0} \right) = r H^0 \partial_r \left(\frac{H^I}{H^0} \right) + \tilde{H}^0 \partial_z \left(\frac{H^I}{H^0} \right) \quad (3.119)$$

which are solved by

$$K^I = \frac{\tilde{H}^0 H^I - H^0 \tilde{H}^I}{(H^0)^2 r^2 + (\tilde{H}^0)^2} \quad (3.120)$$

where \tilde{H}^I are conjugate harmonic functions for H^I . Comparing the quantities $f^{-1}Y_I$ and ξ in both frames then determines the functions K_I, K_0 and the one-form ω in the Toda form of the solution:

$$K_I = H_I - \frac{1}{2}D_{IJK} \left(\frac{H^J H^K}{H^0} - \frac{K^J K^K}{K^0} \right) \quad (3.121)$$

$$K_0 = \Lambda^{GH} + \tilde{H}^0 (H^0)^{-2} L^{GH} + (K^0)^{-1} K^I K_I - \frac{(K^0)^{-2}}{3} D_{IJK} K^I K^J K^K \quad (3.122)$$

$$\omega = \left((H^0)^{-2} L^{GH} + (K^0)^{-2} \left((H^0)^2 r^2 + (\tilde{H}^0)^2 \right)^{-1} \tilde{H}^0 L \right) d\theta_1 \quad (3.123)$$

⁷The H ’s here are just the K ’s of section 3.2.2 in the limit $s_2 \rightarrow 0$, as the ‘Gibbons-Hawking’ frame is the one where we single out the translational Killing vector.

Toric type	$\theta^1 \rightarrow \theta^1 + 2\pi$ monodromy	brane type
IP	$\tau \rightarrow \tau + 2\pi p$	M2 (parabolic)
IH	$\tau \rightarrow e^{2\pi r} \tau$	hyperbolic
IE	$\tau \rightarrow \frac{\cos 2\pi q \tau + \sin 2\pi q}{-\sin 2\pi q \tau + \cos 2\pi q}$	elliptic

Table 4. Relation between toric symmetries, τ monodromies and brane sources. The corresponding profiles for τ , h and μ can be read off from table 3.

where Λ^{GH} is the solution to $d\Lambda^{GH} = \langle d\tilde{H}, H \rangle$. It is an interesting exercise to check consistency by using the above expressions for K in terms of H to see that the equations (3.36)–(3.39) on K reduce to the simple harmonic conditions (3.99) for H .

These solutions, which arose from lifting well-known 4D multicentered solutions to 5D, give rise to a new and as yet unexplored class of 4D solutions when we dimensionally reduce them along the rotational direction. This procedure is a dimensionally reduced version of the ‘9-11 flip’ for type IIA/M theory solutions. It would be interesting to further study the 4D solutions obtained in this manner.

3.3.3 Interpretation of toric solutions as M2 (or exotic) branes

We now proceed to discuss solutions on a toric base with non-constant axidilaton. As we saw in section 3.3.1, when we have toric symmetry the axidilaton is completely fixed by symmetry, see table 3. One can now try to understand the physical interpretation of these profiles for τ . This is most easily done in the case where the Killing vector under which τ transforms non-trivially generates a compact direction. In that case the axidilaton τ has a nontrivial monodromy. Let us first consider the case that the monodromy is of parabolic type, which translates to a charge for the dual 4-form and hence the presence of an M2-brane extended in the 5 external dimensions. This can be generalized to the other classes of monodromy, but their interpretation in terms of basic M-theory objects is less understood, and we will refer to them as exotic branes following [43, 44]. Depending on the monodromy we will speak about hyperbolic or elliptic branes. The parameters p, q, r in table 3 then correspond to brane charges and are quantized in string/M theory. We summarize the relation between toric symmetries, τ monodromies and brane sources in table 4.

In our notation, the Killing vector $k^{(2)} = \partial_{\theta^2}$ leaves τ invariant while $k^{(1)} = \partial_{\theta^1}$ induces a U-duality transformation on τ . Hence in this convention ∂_{θ^2} acts along the worldvolume directions of the brane while ∂_{θ^1} acts in the transverse directions. We would like to interpret at least some of these M2 (or exotic) branes with toric symmetries as a backreaction of these branes in a given background with toric base, such as the ones we reviewed in section 3.3.2. For this interpretation to work the solutions should be such that when taking the limit of zero charge $p, q, r \rightarrow 0$, the metric reduces to the desired background solution. The addition of the brane can preserve the toric symmetries of the background only if the brane is placed

at a fixed point of ∂_{θ^1} in the 3D base; ∂_{θ^1} is then a rotational⁸ symmetry when encircling the brane in the transverse space. Defining $u = |u|e^{i\theta^1}$ to be a local coordinate centered on the fixed point of ∂_{θ^1} , the relation with the coordinate w^1 introduced earlier is

$$u = e^{w^1}, \quad x^1 = \log |u|. \quad (3.124)$$

One point we want to stress is that the solutions so obtained in general do not make sense globally, since τ_2 can become negative in some part of the u -plane. An analogous situation occurs in solutions involving D7 branes, where it is well-known that in order to make a globally well-defined solution one has to combine several such branes [45–47]. This comes at the cost of breaking the rotational symmetry in the transverse plane and hence the toric character of the geometry. The toric solution then describes only the local geometry near one of the branes, as we will illustrate in example 4.2.1. A global non-toric example will be discussed in in example 4.2.2. An interesting loophole in the above argument arises when the locus where τ_2 becomes zero coincides with a boundary of the spacetime. This is in fact what happens in the examples we will discuss in sections 4.2.4 and 4.2.5.

3.3.4 Separated toric solutions

As in the case with one Killing vector, when $s_2 \neq 0$ a more tractable subset of toric solutions is obtained by making a separated ansatz for the Toda potential. The analysis proceeds as in section 3.2.3 with all quantities now independent of $\theta^1 = \text{Im}w^1$. The geometry of the base is completely specified by a function $\Phi(x^1)$ which satisfies an ordinary nonlinear differential equation:

$$\Phi'' - \kappa^2 e^{\mu+s_1x^1-2\Phi} = 0 \quad (3.125)$$

where $\mu(x^1)$ can be read off from table 3. The fact that the problem is reduced to finding a function Φ satisfying an ordinary nonlinear differential equation is what makes this class of solutions (more) tractable and, as we will see below in the examples, it still contains a number of physically interesting solutions. The base metric (3.56) becomes

$$\begin{aligned} ds_4^2 &= K^0(dy_2^2 + gds_2^2) + \frac{1}{K^0} (d\theta^2 + \chi)^2 \\ K^0 &= \frac{g'}{s_2g} \quad \chi = \frac{2}{s_2}\Phi' \\ ds_3^2 &= dy_2^2 + ge^{\mu+s_1x^1-2\Phi} ((dx^1)^2 + (d\theta^1)^2) \end{aligned} \quad (3.126)$$

In our analysis in section 3.2.3 we have already encountered solutions where the 4D base has extra symmetries. In particular, these possess two commuting Killing vectors and are toric. Let us write these solutions more explicitly in the current toric coordinates adapted to both isometries. For constant axidilaton (i.e. class II), $\mu = 0$ and we have the

⁸Note that that this doesn't mean that ∂_{θ^1} must be of the rotational type in our classification, i.e. we can still have $s_1 = 0$.

Class	κ^2	Φ solution	4D symmetry	5D symmetry	Type
II	< 0	(3.127)	$sl(2, \mathbb{R}) \times u(1)$	$sl(2, \mathbb{R}) \times sl(2, \mathbb{R}) \times so(3)$	$AdS_3 \times S^2$
	0	(3.128)	$e(4)$	$iso(1, 4)$	$\mathbb{R}^{1,4}$
	> 0	(3.129)	$so(3) \times u(1)$	$a^2 > 0 : so(4) \times sl(2, \mathbb{R})$ $a^2 < 0 : so(3) \times u(1) \times u(1)$	$AdS_2 \times S^3$ Eguchi-Hanson $\times \mathbb{R}$
IP	< 0	(3.130) generic (3.125)	$sl(2, \mathbb{R}) \times u(1)$ $u(1) \times u(1)$	$sl(2, \mathbb{R}) \times u(1) \times so(3)$ $u(1) \times u(1) \times so(3)$	Godel $\times S^2$ M2 in $AdS_3 \times S^2$
	0	generic (3.125)	$e(2) \times u(1)$	$iso(1, 2) \times u(1)$	M2 in $\mathbb{R}^{1,4}$
	> 0	generic (3.125)	$u(1) \times u(1)$	$a^2 > 0 : u(1) \times u(1) \times so(3)$ $a^2 < 0 : u(1) \times u(1) \times u(1)$	M2 in $AdS_2 \times S^3$ M2 in Eguchi-Hanson $\times \mathbb{R}$

Table 5. Factorized toric solutions and their symmetries. The classes IH, IE are similar to IP but with the M2-brane replaced by an exotic brane.

following solutions to (3.125) (up to some conveniently chosen integration constants):

$$\kappa^2 < 0 : \Phi = \begin{cases} \frac{s_1 x^1}{2} + \ln 2 \sinh \frac{|\kappa| x^1}{2} \\ \frac{s_1 x^1}{2} + \ln |\kappa| x^1 \\ \frac{s_1 x^1}{2} + \ln 2 \sin \frac{|\kappa| x^1}{2} \end{cases} \quad (3.127)$$

$$\kappa^2 = 0 : \quad \Phi = 0 \quad (3.128)$$

$$\kappa^2 > 0 : \quad \Phi = \frac{s_1 x^1}{2} + \ln 2 \cosh \frac{\kappa x^1}{2} \quad (3.129)$$

The different types of solution for $\kappa^2 < 0$ arise because in that case the base has an $sl(2, \mathbb{R})$ symmetry and we can choose $k^{(1)}$ to generate an elliptic, parabolic or hyperbolic isometry respectively. When the axidilaton is turned on (i.e. in classes IP, IH, IE), we saw in (3.60) that there exists special symmetric solution for $\kappa^2 < 0$:

$$\Phi = \frac{1}{2} \ln \left(-\frac{2\kappa^2 e^{3\mu + s_1 x^1}}{3c^2} \right) \quad (3.130)$$

where as before $c = p, q, r$ in classes IP, IH and IE respectively. To check that this in fact solves (3.125) one has to use the property that μ solves a Liouville equation (3.73). The symmetries of the base manifold for these solutions were discussed in section 3.2.3 and we recapitulate them in table 5.

We now discuss the full 5D solutions that we can construct on a toric Kähler base of the factorized form (3.126), following the discussion in section 3.2.3. We distinguish three cases depending on the sign of κ^2 and a^2 . The properties of these 5D solutions are summarized in table 5, anticipating the more detailed discussion in section 4.

$\kappa^2 = 0$. In this case the base has positive definite signature and we can construct a static solution with trivial vector multiplets. The Toda equation is solved by taking $g = s_2 y_2 +$

1, $\Phi = 0$, which leads to

$$ds_5^2 = -dt^2 + e^{\mu+s_1x^1} ((dx^1)^2 + (d\theta^1)^2) + \frac{dy_2^2}{1+s_2y_2} + (1+s_2y_2)(d\theta^2)^2$$

$$F^I = Y^I = 0. \quad (3.131)$$

As we will discuss in more detail in examples 4.1.1, 4.2.1 this type of solution contains flat $\mathbb{R}^{1,4}$ (when the axidilaton is constant) and the near-brane geometry of an M2 or exotic brane in $\mathbb{R}^{1,4}$.

$\kappa^2 > 0, a^2 < 0$. This base also positive definite signature and the static 5D solution looks as follows:

$$ds_5^2 = -dt^2 + \frac{g'}{s_2} e^{\mu+s_1x^1-2\Phi} ((dx^1)^2 + (d\theta^1)^2) + \frac{g'}{s_2g} dy_2^2 + \frac{s_2g}{g'} \left(d\theta^2 + \frac{2}{s_2} \Phi' d\theta^1 \right)^2$$

$$F^I = Y^I = 0$$

$$g = \kappa^2 y_2^2 + 4a^2, \quad \kappa^2 > 0, a^2 < 0 \quad (3.132)$$

with Φ a solution to (3.125). As we will see in examples 4.1.2 and 4.2.5, the solution with constant axidilaton is the $\mathbb{R} \times$ Eguchi-Hanson metric while turning on the axidilaton allows us to describe an M2 or exotic brane in this background.

$\kappa^2 \neq 0, a^2 > 0$. In this case the base is ambipolar and we can construct regular 5D solutions of the form (3.65)–(3.68) for the toric case, leading to

$$ds_5^2 = \left(\frac{p^3}{6} \right)^{2/3} \left[\frac{dy_2^2}{g} - \frac{g}{4a^2\kappa^2} \alpha^2 + \frac{1}{\kappa^2} \left(\alpha + \Phi' d\theta^1 + \frac{s_2}{2} d\theta^2 \right)^2 \right. \\ \left. + e^{\mu+s_1x^1-2\Phi} ((dx^1)^2 + (d\theta^1)^2) \right]$$

$$\alpha = -\frac{bs_2}{2} d\theta^2 - \frac{24a\kappa^2}{s_2p^3} \left(dt - \frac{R}{2} d\theta^1 \right) \quad (3.133)$$

$$F^I = \frac{p^I}{2a} dy_2 \wedge \alpha, \quad Y^I = \left(\frac{6}{p^3} \right)^{\frac{1}{3}} p^I$$

$$g = \kappa^2 y_2^2 + 4a^2, \quad \kappa^2 \neq 0, a^2 > 0$$

$$0 = \Phi_{x^1x^1} - \kappa^2 e^{\mu+s_1x^1-2\Phi}. \quad (3.134)$$

Here we have chosen the closed one-form in (3.68) to be proportional to $d\theta^1$: $\lambda = -Rd\theta^1/2$. As we will illustrate in examples 4.1.4), (4.1.3), (4.2.4 and 4.2.5, these solutions include the $\text{AdS}_3 \times \text{S}^2$ and $\text{AdS}_2 \times \text{S}^3$ backgrounds as well as M2/exotic branes added to them.

Let us now also discuss how to determine the constraint on the parameters that we have to impose in order for ω to satisfy the integrability condition (3.47) or, in other words,

for it to be free of Dirac string singularities. Recall from (3.65) that for the current class of solutions ω is given by

$$\omega = - \left(\frac{s_2 b p^3}{12 a \kappa^2} \Phi' + R \right) d\theta^1 \quad (3.135)$$

If the range the coordinate is such that $|x^1|$ can become large, the Liouville-like equation (3.125) is compatible with an asymptotically linear behaviour of Φ for large $|x^1|$:

$$\Phi \sim m|x^1| \quad (3.136)$$

for some constant m large enough that the second in term (3.125) vanishes for large $|x^1|$. Hence in the 3D metric (3.126), ∂_{θ^1} has a fixed line for $|x^1| \rightarrow \infty$, where there is a coordinate singularity. The condition on ω is then that the coefficient of $d\theta^1$ should vanish for large $|x^1|$, so that we should impose

$$a = -(\text{sgn} x^1) \frac{s_2 b p^3 m}{12 R \kappa^2} \quad (3.137)$$

We observe from (3.133) that for $\kappa^2 < 0$ this also removes closed timelike curves (CTCs) which would otherwise appear near the fixed line $|x^1| \rightarrow \infty$. As we will illustrate in example 4.2.4, it is still possible for CTCs to crop up elsewhere in the spacetime, arising from having a lot of rotating matter. This is what happens in the classic example of the Gödel universe [48], of which we will encounter a supersymmetrized version.

4 Examples

In this section we will discuss some concrete examples of solutions with a toric Kähler base. We begin by reviewing solutions with constant axidilaton where the 4D base is toric hyperkähler, illustrating how the simplest solutions constructed from axially symmetric Gibbons-Hawking bases as in section 3.3.2 can also be obtained from separated solutions of the Toda equation of section 3.3.4. We then turn to solutions with axidilaton, focussing on those solutions which describe backreacted M2 and exotic branes placed in a background with toric base. We will discuss in detail backreacted branes in flat space and the highly symmetric Gödel \times S² solution which, as we will argue, arises from a distribution of branes in the AdS₃ \times S² background. We will also comment on the solutions describing individual branes in the AdS₃ \times S², Eguchi-Hanson and AdS₂ \times S³ backgrounds, which will be discussed in more detail in a separate publication.

4.1 Solutions with toric hyperkähler base

We have discussed two ways of constructing such solutions: from axisymmetric solutions with Gibbons-Hawking base in section 3.3.2 and from solving the Toda equation with a separated ansatz in section 3.3.4. We will see that examples of the first method with up to two Gibbons-Hawking centers fit precisely in the second type of solutions with one translational and one rotational Killing vector.

4.1.1 Flat spacetime

To get a feeling for the definitions and coordinate systems introduced above, let's warm up by seeing how the flat $\mathbb{R}^{1,4}$ background fits in our formalism. The simplest solution to (3.74) with $\mu = 0$ is of the separated form

$$\mathcal{K} = \frac{e^{s_1 x^1} - s_1 x^1 - 1}{s_1^2} + \frac{e^{s_2 x^2} - s_2 x^2 - 1}{s_2^2} \quad (4.1)$$

The linear terms, which could be removed by a Kähler transformation, are added to have a well-behaved $s_i \rightarrow 0$ limit. The corresponding 5D solution with trivial vector multiplets gives the Minkowski metric in the form

$$ds_5^2 = -dt^2 + e^{s_1 x^1} ((dx^1)^2 + (d\theta^1)^2) + e^{s_2 x^2} ((dx^2)^2 + (d\theta^2)^2). \quad (4.2)$$

This example illustrates the origin of the terminology of translational and rotational Killing vectors: for $s_i \rightarrow 0$, the Killing vector ∂_{θ^i} generates a Minkowski translation while for $s_i > 0$ it generates a rotation around the fixed point of ∂_{θ^i} which is at $x^i \rightarrow -\infty$. For the minimal quantum $s_i = 2$ the space is free of conical singularities⁹ while $s_i = 2N$ gives a \mathbb{Z}_N orbifold singularity in the fixed point.

From (3.77) we find the symplectic coordinates y_i :

$$x^1 = \frac{\ln(1 + s_1 y_1)}{s_1}, \quad x^2 = \frac{\ln(1 + s_2 y_2)}{s_2}. \quad (4.3)$$

In particular, the Toda potential is

$$\Psi = x^2 = \frac{\ln(1 + s_2 y_2)}{s_2} \quad (4.4)$$

which corresponds to a factorized solution (3.54) of the Toda equation (3.82) with $g = s_2 y_2 + 1$ and $\Phi = 0$. In the Toda frame, the metric is precisely (3.131) with $\mu = 0$.

Making the Legendre transform (3.78) we find the symplectic potential

$$\mathcal{S} = \frac{(1 + s_1 y_1) \log(1 + s_1 y_1) - s_1 y_1}{s_1^2} + \frac{(1 + s_2 y_2) \log(1 + s_2 y_2) - s_2 y_2}{s_2^2} \quad (4.5)$$

which indeed satisfies the Monge-Ampère-type equation (3.81). From the form of the metric in symplectic coordinates

$$ds_5^2 = -dt^2 + \frac{dy_1^2}{1 + s_1 y_1} + \frac{dy_2^2}{1 + s_2 y_2} + (1 + s_1 y_1)(d\theta^1)^2 + (1 + s_2 y_2)(d\theta^2)^2 \quad (4.6)$$

one sees that the moment polytope of flat space is

$$y_1 \geq -\frac{1}{s_1}, \quad y_2 \geq -\frac{1}{s_2}. \quad (4.7)$$

For $s_1 = 0, s_2 = 2$ we can alternatively describe the solution in terms of an axially symmetric Gibbons Hawking base (3.110)–(3.116) where

$$H^0 = 1, \quad r = \sqrt{1 + 2y_2} = e^{x^2}, \quad z = x^1 = y^1 \quad (4.8)$$

and all the other quantities $\tilde{H}^0 = H^I = H_I = H_0 = \omega^{GH}$ are taken to be zero. As a check one easily verifies that the relations (3.94) are indeed satisfied.

⁹Recall that, in case the θ^i is a compact coordinate, we have chosen its period to be 2π .

4.1.2 Two-center Taub-NUT: Eguchi-Hanson

Next, let's consider a Gibbons-Hawking base with two centers of equal charge at a distance $2b$ apart and without constant term in H^0 (i.e. ALE):

$$H^0 = \frac{P}{2\sqrt{r^2 + (z-b)^2}} + \frac{P}{2\sqrt{r^2 + (z+b)^2}} \quad (4.9)$$

As we will see, P is the NUT charge of the solution (or the D6-charge after dimensional reduction along the translational direction), and for $P = 1$ the metric on the base is the Eguchi-Hanson metric [49]. Since this is axially symmetric we can rewrite it in a toric form with one rotational and one translational Killing vector. To do this we first work out the relation between complex and symplectic coordinates using (3.106)–(3.109):

$$x^1 = P \operatorname{arctanh} \frac{Py_1}{y_2} \quad (4.10)$$

$$x^2 = \ln \frac{\sqrt{(y_2^2 - b^2 P^2)(y_2^2 - P^2 y_1^2)}}{Py_2} \quad (4.11)$$

The Toda potential can be computed from (3.103) and one finds that it is of the factorized form (3.54) with positive $\kappa^2 = 4/P^2$:

$$e^\Phi = 2 \cosh \frac{x_1}{P} \quad g = 4 \left(\frac{y_2^2}{P^2} - b^2 \right) \quad (4.12)$$

From our discussion in section 3.2.3 we know that the solution has $so(3) \times u(1)$ isometry, which is indeed a well-known property of the Eguchi-Hanson metric. The Kähler potential is given by (3.64), which can be brought into the more standard form which makes the $so(3) \times u(1)$ symmetry manifest by making a holomorphic coordinate transformation

$$u^\pm = \frac{1}{\sqrt{2}} e^{\frac{1}{2}(w^2 \pm \frac{w^1}{P})} \quad (4.13)$$

The Kähler potential then becomes

$$\mathcal{K} = bP \left(\sqrt{1 + \frac{(u^+ \bar{u}^+ + u^- \bar{u}^-)^2}{b^2}} - \operatorname{arctanh} \sqrt{1 + \frac{(u^+ \bar{u}^+ + u^- \bar{u}^-)^2}{b^2}} \right) \quad (4.14)$$

We can also compute the symplectic potential from (3.103):

$$\mathcal{S} = y_2 \ln \frac{\sqrt{(y_2^2 - b^2 P^2)(y_2^2 - P^2 y_1^2)}}{y_2 P} - bP \operatorname{arctanh} \frac{bP}{y_2} + y_1 P \operatorname{arctanh} \frac{Py_1}{y_2} - y_2$$

The corresponding moment polytope is the following region in \mathbb{R}^2 (see figure 1(b)):

$$|y_2| \geq Pb, \quad |y_2| \geq P|y_1|. \quad (4.15)$$

It's convenient to introduce prolate spheroidal coordinates:

$$r = b \sinh \rho \sin \eta \quad (4.16)$$

$$z = b \cosh \rho \cos \eta \quad (4.17)$$



Figure 1. The moment polytopes for (a) flat space with $s_1 = 0, s_2 \neq 0$, (b) the Eguchi-Hanson metric, (c) the ambipolar continued Eguchi-Hanson metric, and (d) the ambipolar metric for a Taub-NUT anti-Taub-NUT system.

in terms of which x^1, y_2 are

$$x^1 = P \ln \cot \frac{\eta}{2} \quad (4.18)$$

$$y_2 = bP \cosh \rho. \quad (4.19)$$

The static 5D metric constructed from this 4D base is then (3.132) with $s_1 = 0, s_2 = 2, \kappa^2 = 4/P^2, a^2 = -b^2$ and takes the form

$$ds_5^2 = -dt^2 + bP \cosh \rho \left(d\eta^2 + \frac{1}{P^2} \sin^2 \eta (d\theta^1)^2 + d\rho^2 + \tanh^2 \rho \left(d\theta^2 + \frac{1}{P} \cos \eta d\theta^1 \right)^2 \right) \quad (4.20)$$

Note the manifest $so(3) \times u(1) \times u(1)$ symmetry. For generic value of P , there will be a conical singularity at $\rho = 0$, but for $P = 1$ we obtain the smooth Eguchi-Hanson manifold. In the limit $b \rightarrow 0$, the solution reduces to the geometry near a single centered Taub-NUT charge P , which for $P = 1$ is equivalent to flat spacetime.

4.1.3 The $AdS_2 \times S^3$ solution

As a first example of a solution with an ambipolar base, consider the base of the previous example upon the analytic continuation $b \rightarrow ia$. For $P = 1$ the resulting metric was already discussed in [49] where it was called ‘type II’. The Toda potential is of the factorized form (3.54) with with positive $\kappa^2 = 4/P^2$ and

$$e^\Phi = 2 \cosh \frac{x_1}{P} \quad g = 4 \left(\frac{y_2^2}{P^2} + a^2 \right). \quad (4.21)$$

The Kähler potential is, in the variables (4.13):

$$\mathcal{K} = aP \left(\sqrt{\frac{(u^+ \bar{u}^+ + u^- \bar{u}^-)^2}{a^2} - 1} - \arctan \sqrt{\frac{(u^+ \bar{u}^+ + u^- \bar{u}^-)^2}{a^2} - 1} \right) \quad (4.22)$$

which shows the $so(3) \times u(1)$ isometry discussed in section 3.2.3. The symplectic potential is

$$\mathcal{S} = y_2 \ln \frac{\sqrt{(y_2^2 + a^2 P^2)(y_2^2 - P^2 y_1^2)}}{y_2 P} - aP \operatorname{Parctan} \frac{aP}{y_2} + y_1 P \operatorname{Parctanh} \frac{Py_1}{y_2} - y_2. \quad (4.23)$$

The image of the moment map is a conical region in \mathbb{R}^2 (see figure 1(c)):

$$|y_2| \leq P|y_1| \quad (4.24)$$

This ambipolar metric is therefore characterized by a generalized moment polytope which consists of two convex regions, the upper and lower parts of the cone, whose tips touch at $y_1 = y_2 = 0$, where the base becomes singular.

After the change of coordinates

$$x^1 = P \ln \cot \frac{\eta}{2} \quad (4.25)$$

$$y_2 = bP \sinh \rho. \quad (4.26)$$

the metric on the 4D base is manifestly ambipolar:

$$ds_4^2 = aP \sinh \rho \left(d\eta^2 + \frac{1}{P^2} \sin^2 \eta (d\theta^1)^2 + d\rho^2 + \coth^2 \rho \left(d\theta^2 + \frac{1}{P} \cos \eta d\theta^1 \right)^2 \right). \quad (4.27)$$

The fact that the base is ambipolar means that we will have to turn on vector multiplets to get a regular 5D solution. We take the solution of the form (3.133), where in the present example we should take $s_1 = 0, s_2 = 2, k^2 = 1/Q^2, a^2 > 0$ and Φ given by (4.21). Taking furthermore the parameters $b = R = 0$ in (3.133) gives $\omega = 0$ and hence no further constraints have to be imposed on the parameters from regularity of ω . The resulting solution is

$$ds_5^2 = \frac{P^2}{4} \left(\frac{p^3}{6} \right)^{\frac{2}{3}} \left[- \left(\frac{24a \cosh \rho}{p^3 P} \right)^2 dt^2 + d\rho^2 + d\eta^2 + d \left(\frac{\theta^1}{P} \right)^2 + d \left(\theta^2 - \frac{48a}{p^3 P^2} t \right)^2 + 2 \cos \eta d \left(\frac{\theta^1}{P} \right) d \left(\theta^2 - \frac{48a}{p^3 P^2} t \right) \right] \quad (4.28)$$

$$F^I = -p^I \left(\frac{24a \cosh \rho}{p^3 P} \right) dt \wedge d\rho, \quad Y^I = \left(\frac{6}{p^3} \right)^{\frac{1}{3}} p^I \quad (4.29)$$

The metric is locally $\text{AdS}_2 \times S^3$ with a spinning S^3 and with flux on AdS_2 . This solution arises as the near-horizon limit of charged BPS black holes.

4.1.4 TN-anti-TN with flux

An important solution with ambipolar base comes from a Gibbons-Hawking metric with two oppositely charged centers:

$$H^0 = \frac{P}{2\sqrt{r^2 + (z+a)^2}} - \frac{P}{2\sqrt{r^2 + (z-a)^2}} \quad (4.30)$$

Working out the relation between complex and symplectic coordinates one finds

$$x^1 = P \text{arccoth} \frac{Py_1}{y_2} \quad (4.31)$$

$$x^2 = \ln \frac{\sqrt{(y_2^2 - a^2 P^2)(y_2^2 - P^2 y_1^2)}}{Py_2} \quad (4.32)$$

The Toda potential Ψ is of the factorized form (3.54) with negative $\kappa^2 = -4/P^2$:

$$e^\Phi = 2 \sinh \frac{x_1}{P}, \quad g = 4 \left(a^2 - \frac{y_2^2}{P^2} \right) \quad (4.33)$$

so that we know from the discussion in section 3.2.3 that the 4D base has $sl(2) \times u(1)$ isometry. The Kahler potential is, in the variables (4.13):

$$\mathcal{K} = aP \left(\sqrt{1 - \frac{(u^+ \bar{u}^+ - u^- \bar{u}^-)^2}{a^2}} - \operatorname{arctanh} \sqrt{1 - \frac{(u^+ \bar{u}^+ - u^- \bar{u}^-)^2}{a^2}} \right). \quad (4.34)$$

For the symplectic potential one finds

$$\mathcal{S} = y_2 \ln \frac{\sqrt{(y_2^2 - a^2 P^2)(y_2^2 - P^2 y_1^2)}}{P y_2} - aP \operatorname{arctanh} \frac{aP}{y_2} - P y_1 \operatorname{arctanh} \frac{P y_1}{y_2} - y_2$$

The image of the moment map is in this case the region

$$|y_2| \leq aP, \quad |y_2| \leq P|y_1| \quad (4.35)$$

which once again consists of two convex regions touching in the point $y_1 = y_2 = 0$ where the base is singular, see figure 1(d).

It is once again convenient to switch to prolate spheroidal coordinates (4.17) in terms of which

$$x^1 = P \ln \coth \frac{\rho}{2} \quad (4.36)$$

$$y_2 = aP \cos \eta \quad (4.37)$$

and the ambipolar base metric is manifestly $sl(2, \mathbb{R}) \times u(1)$ symmetric:

$$ds_4^2 = -aP \cos \eta \left(d\rho^2 + \frac{1}{P^2} \sinh^2 \rho (d\theta^1)^2 + d\eta^2 + \tan^2 \eta \left(d\theta^2 + \frac{1}{P} \cosh \rho d\theta^1 \right)^2 \right) \quad (4.38)$$

We can construct a regular 5D solution on this base by turning on appropriate fluxes. Indeed, it arises from lifting a D6-anti D6 system which can be made stable only if suitable worldvolume fluxes are turned on, providing additional repulsive forces. In the Gibbons-Hawking form, the relevant solution is of the form (3.110)–(3.116) with harmonic functions [8, 35]:

$$H^I = \frac{p^I P}{4} \left(\frac{1}{\sqrt{r^2 + (z+a)^2}} + \frac{1}{\sqrt{r^2 + (z-a)^2}} \right) = -\frac{p^I P}{2a} \frac{\cos \eta}{\cosh^2 \rho - \cos^2 \eta} \quad (4.39)$$

$$H_I = \frac{p_I P}{16} \left(\frac{1}{\sqrt{r^2 + (z+a)^2}} - \frac{1}{\sqrt{r^2 + (z-a)^2}} \right) = \frac{p_I P}{8a} \frac{\cosh \rho}{\cosh^2 \rho - \cos^2 \eta} \quad (4.40)$$

$$H_0 = \frac{p^3 P}{96} \left(\frac{1}{\sqrt{r^2 + (z+a)^2}} + \frac{1}{\sqrt{r^2 + (z-a)^2}} \right) - R = \frac{p^3 P}{48a} \frac{\cos \eta}{\cosh^2 \rho - \cos^2 \eta} - R \quad (4.41)$$

where the parameter R is related to the asymptotic radius of the M-theory circle.

Rewriting the solution in the Toda form using the formulas (3.120)–(3.123), one finds that the functions K for this solution are precisely of the form (3.65) where the parameter $b = 1$. It is also instructive to see how the constraint on the distance $2a$ between the centers arises in both frames. In the Gibbons-Hawking frame, the constraint is (3.117) which gives

$$a = \frac{p^3 P}{24R}. \quad (4.42)$$

From the point of view of the Toda frame, the same constraint follows from equation (3.137) imposing absence of singularities in ω . From the $x^1 \rightarrow \infty$ behavior of Φ , we see that the constant m in (3.136) is $m = \frac{1}{P}$, leading once again to (4.42).

Using (4.42) to eliminate a , we obtain the full 5D solution

$$ds_5^2 = \frac{P^2}{4} \left(\frac{p^3}{6} \right)^{\frac{2}{3}} \left[- \left(\frac{2dt}{PR} + (\cosh \rho - 1) \frac{d\theta^1}{P} \right)^2 + d\rho^2 + \frac{1}{P^2} \sinh^2 \rho (d\theta^1)^2 + d\eta^2 + \sin^2 \eta d \left(\theta^2 + \frac{\theta^1}{P} - \frac{2t}{PR} \right)^2 \right] \quad (4.43)$$

$$F^1 = -\frac{p^I P}{2} \sin \eta d\eta \wedge d \left(\theta^2 + \frac{\theta^1}{P} - \frac{2t}{PR} \right), \quad Y^I = \left(\frac{6}{p^3} \right)^{\frac{1}{3}} p^I \quad (4.44)$$

We again obtain a smooth solution for $P = 1$, corresponding to precisely one unit of NUT charge at each of the centers. The first part of the metric is then global AdS_3 in rotating coordinates, written as a fibration over the hyperbolic plane. The second part is a (nontrivially fibered) round S^2 or a ‘bubble’ in the language of [28].

4.2 Solutions with toric Kähler base

In the remainder of this section we will consider examples where the axidilaton is turned on and which have a toric Kähler base. Following the general discussion in section 3.3.3 we will focus on solutions which describe the backreaction of M2- or exotic branes placed in one of the backgrounds with hyperkähler base that were discussed in the previous section. All these solutions fit within the separated toric ansatz discussed in section 3.3.4.

4.2.1 Local M2-branes in flat space

We start by studying the solutions describing the local geometry near M2- or exotic branes in flat space following the procedure described in section 3.3.3. Consider flat space parameterized as in (4.2) with $s_1 = 2$ and $s_2 = 0$, so that the Killing vector ∂_{θ^1} is rotational with $s_1 = 2$ while ∂_{θ^2} is translational. The Killing vector ∂_{θ^1} has a fixed locus of codimension 2 at $x^1 = -\infty$ and, as discussed in section 3.3.3, we can backreact an M2- or exotic brane there without spoiling the toric symmetry. The resulting solution is given by (3.131) with $s_1 = 2$ and $s_2 = 0$. The Toda potential is simply $\Psi = y_2$ and transforming to complex coordinates one finds the Kähler potential $\mathcal{K} = M(x^1) + \frac{(x^2)^2}{2}$ where the function M should satisfy

$$M'' = e^{\mu+2x^1}. \quad (4.45)$$

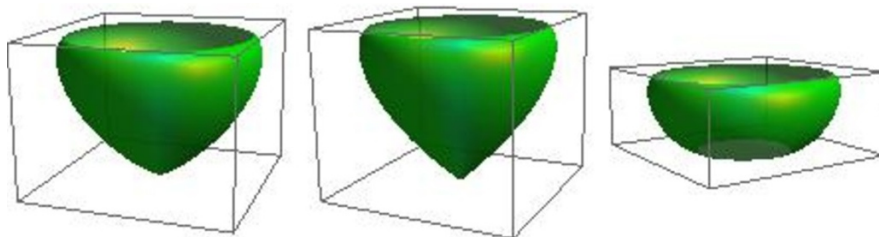


Figure 2. The geometry transverse to (a) an M2-brane, (b) an elliptic brane and (c) a hyperbolic brane, illustrated as a (local) isometric embedding in \mathbb{R}^3 .

The full solution is

$$ds_5^2 = -dt^2 + e^{\mu(x^1)+2x^1} dw^1 d\bar{w}^1 + dw^2 d\bar{w}^2 \quad (4.46)$$

$$= -dt^2 + e^{\mu(\ln|u|)} dud\bar{u} + dw^2 d\bar{w}^2 \quad (4.47)$$

where in the second line we have switched to the local coordinate (3.124) $u = e^{w^1}$ centered on the brane position. The curvature $R^{(2)}$ of the transverse space and the accumulated deficit angle δ as a function of $|u|$ are given in terms of μ by

$$R^{(2)} = \frac{c^2 e^{-3\mu(\ln|u|)}}{|u|^2}, \quad c = p, q, r \quad (4.48)$$

$$\delta(|u|) = -\pi\mu'(\ln|u|) \quad (4.49)$$

We will now discuss the resulting local geometry near each type of brane in turn.

For the M2 brane, which has parabolic monodromy $\tau \rightarrow \tau + 2\pi q$ under $u \rightarrow e^{2\pi i}u$, we obtain

$$\tau(u) = i(1 - p \ln u); \quad h(u) = 0 \quad (4.50)$$

$$ds_5^2 = -dt^2 + (1 - p \ln|u|) dud\bar{u} + dw^2 d\bar{w}^2. \quad (4.51)$$

We see from (4.49) that there is a mild, integrable, curvature singularity at the brane position $u = 0$, although there is no conical deficit there as was observed in [46, 47]. There is a further singularity at $|u| = e^{1/p}$ where also τ_2 vanishes and the solution breaks down. This illustrates the fact that the solution with only a single M2-brane does not make sense globally and other objects need to be introduced to make the solution well-behaved. We will illustrate how to do this in the next section. The geometry of the space transverse to the M2 brane, isometrically embedded in \mathbb{R}^3 , is illustrated in figure 2(a).

For the elliptic brane, which has monodromy $\tau \rightarrow \frac{\cos 2\pi q\tau + \sin 2\pi q}{-\sin 2\pi q\tau + \cos 2\pi q}$ under $u \rightarrow e^{2\pi i}u$, the local geometry is

$$\tau(u) = i \tanh(1 - q \ln u); \quad h(u) = 2 \ln \cosh(1 - q \ln u) \quad (4.52)$$

$$ds_5^2 = -dt^2 + \frac{1}{2} \sinh 2(1 - q \ln|u|) dud\bar{u} + dw^2 d\bar{w}^2 \quad (4.53)$$

One finds that such a brane produces a conical singularity with deficit angle $\delta(0) = 2\pi q$ at the brane position, see figure 2(b). Once again there is a curvature singularity at $|u| = e^{1/p}$ where the local solution breaks down.

The hyperbolic brane has monodromy $\tau \rightarrow e^{2\pi r} \tau$ and leads to the local geometry

$$\tau = iu^{-ir}; \quad h = ir \ln u \quad (4.54)$$

$$ds_5^2 = -dt^2 + \cos r \ln |u| dud\bar{u} + dw^2 d\bar{w}^2 \quad (4.55)$$

Since the solution is periodic in $\ln |u|$ it is problematic to interpret it as arising from a single brane source at $u = 0$.¹⁰ It is natural to restrict $|u|$ to a single period

$$e^{-\frac{\pi}{2r}} \leq |u| \leq e^{\frac{\pi}{2r}}; \quad (4.56)$$

since there are curvature singularities at both ends of the interval. See figure 2(c).

As an extra check we see that we indeed recover the flat metric when we let the charges p, q, r go to zero (or equivalently, let u approach $|u| = 1$). We recognize in these formulas the 5D versions of the local backreacted ‘Q-brane’ solutions in flat space of [46].

4.2.2 Global M2-brane solutions

We saw that solutions containing a single M2 or exotic brane in flat space do not make sense globally as τ_2 becomes negative in some region. This is completely analogous to what happens for D7 brane solutions in type IIB/F theory, and in that context it is well-known how to remedy the problem and construct solutions which do make sense globally [45]. Such solutions always contain several branes and therefore inevitably break the rotational invariance in the transverse space [47] and are therefore no longer toric. Let us illustrate this with a simple example, we refer to [45, 46] for more general solutions.

We describe here a simple global solution which contains an M2-brane at $u = 0$. Since we would like u to run over the complex plane and τ to take values in the fundamental region in the upper half plane, we use Klein’s modular invariant j -function to map the fundamental region into the complex plane:

$$j(\tau(u)) = 1 + \frac{1}{u} \quad (4.57)$$

Near $u = 0$, this behaves as

$$\tau \sim \frac{1}{2\pi i} \ln u + \text{regular terms} \quad (4.58)$$

so that we indeed have monodromy $\tau \rightarrow \tau + 1$ when encircling $u = 0$. Near $u = \infty$ we have

$$\tau \sim i. \quad (4.59)$$

To obtain a modular invariant metric which is nondegenerate in $u = 0$, we choose the function h as:

$$h = 4 \ln \left(\eta(\tau(u)) u^{-1/24} \right). \quad (4.60)$$

The metric then becomes

$$ds_5^2 = -dt^2 + \frac{\tau_2(u) |\eta(\tau(u))|^4}{|u|^{1/6}} dud\bar{u} + dw^2 d\bar{w}^2 \quad (4.61)$$

¹⁰In type IIB/F-theory, the status of the object with hyperbolic monodromy is similarly unclear [46].

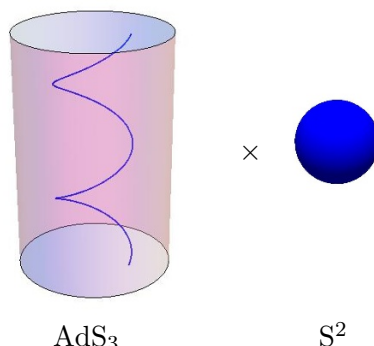


Figure 3. A brane probe at constant w^1 in the $\text{AdS}_3 \times \text{S}^2$ background wraps the S^2 and spirals around in AdS_3 .

It's easy to see that it satisfies the condition (2.31) for being supersymmetric. At large u , there is a deficit angle of $\frac{2\pi}{12}$. This solution contains, besides the M2-brane at $u = 0$, two elliptic branes at the points where $\tau = i$ and $\tau = e^{2\pi i/3}$ with charges $q = 1/4$ and $q = 1/6$ respectively. Although rotational invariance in the u -plane is broken, one can check that the solution indeed takes the toric form (4.51), (4.53) near the positions of the various branes.

4.2.3 Branes in the TN-anti-TN background

Next we would like to study the backreaction of an M2-brane (or one of its exotic cousins) in the TN-anti-TN background of example 4.1.4. As we discussed there, this background is simply global AdS_3 with a round S^2 fibered over it. Now consider the submanifold $w^1 = \text{constant}$ in this background, in other words $\rho = \rho_0, \theta^1 = \theta_0^1$ in the coordinates (4.17). This is a holomorphic surface within the hyperkähler base, and hence a probe M2 or exotic brane placed at this locus will preserve supersymmetry [8]. Such a BPS brane wraps the S^2 and is static in AdS_3 with respect to the time coordinate in (4.43). Nevertheless, because the metric is not static but only stationary in these coordinates, it carries angular momentum J_{θ^1} proportional to $\sinh^2 \frac{\rho_0}{2}$. Indeed, in terms of standard global coordinates such a brane spirals around at a distance ρ_0 from the center of AdS_3 , see figure 3. Introducing the brane breaks the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SO}(3)$ symmetry of the background down to the subgroup

$$\text{U}(1) \times \text{U}(1) \times \text{SO}(3). \quad (4.62)$$

From the point of view of the 4D theory obtained by dimensionally reducing along θ^1 , the M2-brane becomes a D2 brane wrapping an ellipsoidal surface with a D6 and anti-D6 brane at its focal points, see figure 4(a). The angular momentum along θ^1 becomes D0-brane charge sourced by worldvolume flux. This brane configuration carries the same charges as a D0-D4 black hole and was conjectured to play the role of a microstate geometry for this black hole in the black hole deconstruction proposal of [8]. An argument from quantizing the probe worldvolume theory also suggests that these microstates are sufficiently typical to account for the leading contribution to the black hole entropy in a specific large charge limit [51].

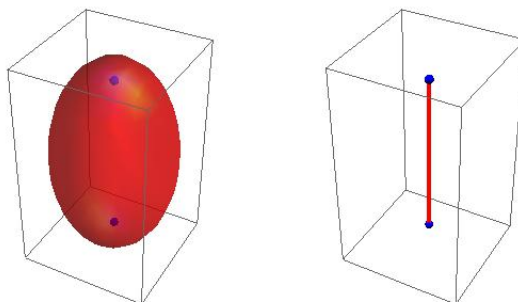


Figure 4. (a) From the 4D point of view, a brane probe at constant w^1 is an ellipsoidal D2-brane surrounding a D6 and anti-D6 brane. (b) The brane probe at the center of AdS_3 is a collapsed ellipsoidal D2-brane in 4D.

Let's focus for simplicity on a brane inserted at the 'center' of AdS_3 , meaning at $\rho_0 = 0$. Within the 3D base this is the fixed locus of the Killing vector ∂_{θ^1} , and we know from the discussion in section 3.3.3 that the backreacted solution will still have a toric base with Killing vectors¹¹ $\partial_{\theta^1}, \partial_{\theta^2}$, so let's attempt to find this solution. From the 4D point of view, the brane at the center of AdS represents a collapsed ellipsoidal brane, see figure 4(b).

Since the background is a solution of the factorized form (3.133)–(3.134) with $\kappa^2 = -4/P^2, s_1 = 0, s_2 = 2, a^2 > 0, b = 1$, it is natural to also look for the backreacted solution within the factorized ansatz with these values of the parameters. Although we don't have a definite proof that the backreacted solution must remain in the factorized form, we expect that it should since the brane leaves the $SO(3)$ symmetry unbroken and the factorized ansatz naturally leads to solutions with $SO(3)$ invariance as discussed below (3.134). Under this assumption, the solution will be of the form (3.133)–(3.134) with the aforementioned values of the parameters and with $\Phi(x^1)$ a solution of

$$\Phi'' + \frac{4}{P^2} e^{\mu-2\Phi} = 0 \quad (4.63)$$

where $\mu(x^1)$ takes one of the three forms given in table 3 depending on whether the brane is of parabolic (M2), hyperbolic or elliptic type.

4.2.4 Gödel $\times S^2$ and its brane interpretation

We know already one particular solution to (4.63), namely (3.130):

$$e^{2\Phi} = \frac{8e^{3\mu}}{3c^2 P^2}. \quad (4.64)$$

which we argued to lead to a highly symmetric 5D solution of the form (3.133)–(3.134). This solution is hence a first candidate for a backreacted brane in $AdS_3 \times S^2$. This was the guess made in [9] where, as we shall presently review, it was also shown to have several

¹¹Note that the choice $\rho = 0$ is merely convenience, since because of homogeneity any of the worldlines $\rho = \rho_0, \theta^1 = \theta_0^1$ discussed above are the fixed locus of some Killing vector $\partial_{\tilde{\theta}^1}$, and inserting a brane there will still give rise to a toric configuration with respect to $\partial_{\tilde{\theta}^1}, \partial_{\theta^2}$.

problematic properties which rule it out as the backreaction of a single brane. Nevertheless it is clear that the solution must have some interpretation in terms of branes in $\text{AdS}_3 \times \text{S}^2$ and we will now give such an interpretation.

Let's first discuss the solution expected to represent an elliptic brane, where we take

$$e^\mu = \frac{1}{2} \sinh 2(1 - qx^1). \quad (4.65)$$

This is perhaps the most symmetric situation since ∂_{θ^1} generates an elliptic isometry within the $\text{SL}(2, \mathbb{R})$ symmetry of the 3D metric, so it's natural to first consider the case where τ has an elliptic monodromy as well. Plugging into (4.64) leads to a base of the form (3.126) which is a toric Kähler manifold of the ambipolar type which as far as we know has not yet appeared in the literature. The Kähler potential can be read off from (3.64):

$$\mathcal{K} = aP \left(\sqrt{1 - \frac{2e^{2x^2+3\mu}}{3a^2q^2P^2}} - \arctan \sqrt{1 - \frac{2e^{2x^2+3\mu}}{3a^2q^2P^2}} \right). \quad (4.66)$$

Making a Legendre transformation one finds the symplectic potential

$$\mathcal{S} = y_2 \ln \frac{\sqrt{(a^2P^2 - y_2^2)(y_1^2 - 9q^2y_2^2)^{3/2}}}{qy_2^3} - aP \operatorname{arctanh} \frac{aP}{y_2} - \frac{y_1}{2q} \operatorname{arctanh} \frac{y_1}{3qy_2} - y_2 + \frac{y_1}{q}$$

The moment polytope is given by

$$|y_2| \leq aP, \quad |y_2| \leq \frac{1}{3q} |y_1| \quad (4.67)$$

It is of the same shape as the polytope (4.35) governing the $\text{AdS}_3 \times \text{S}^2$ solution depicted in figure 1(d); hence this particular polytope admits both a toric hyperkähler as well as a toric Kähler metric.

Switching to coordinates η, ρ

$$x^1 = \frac{1}{q} \left(1 - \frac{1}{2} \ln \tanh \frac{\rho}{2} \right) \quad (4.68)$$

$$y_2 = aP \cos \eta \quad (4.69)$$

the ambipolar metric on the 4D base is:

$$ds_4^2 = -aP \cos \eta \left(\frac{3}{2} (d\rho^2 + 4q^2 \sinh^2 \rho (d\theta^1)^2) + d\eta^2 + \tan^2 \eta (d\theta^2 + 3q \cosh \rho d\theta^1)^2 \right) \quad (4.70)$$

We note something interesting: taking θ^1 to have period 2π as before, the 3D base has a conical defect singularity for generic charge q , but it becomes smooth for

$$q = \frac{1}{2}. \quad (4.71)$$

We will take the brane charge to have this value and will comment on its interpretation below.

Now we turn to the 5D solution (3.133)–(3.134). The parameter a is again fixed by requiring that $d\omega$ doesn't have Dirac string singularities, or equivalently, requiring absence of CTCs near $x^1 \rightarrow \infty$. From (4.64) we read off the asymptotic behaviour of Φ , $\Phi \sim 3qx^1$, so that in (3.137) we have to take $m = 3q$ and impose the following relation between the parameters

$$a = \frac{p^3 q P^2}{8R}. \quad (4.72)$$

The full 5D solution then reads

$$\tau = i \tanh \left(1 - \frac{1}{2} w^1 \right) = - \frac{\cos \frac{\theta^1}{2} + i e^\rho \sin \frac{\theta^1}{2}}{\sin \frac{\theta^1}{2} - i e^\rho \cos \frac{\theta^1}{2}} \quad (4.73)$$

$$ds_5^2 = \frac{P^2}{4} \left(\frac{p^3}{6} \right)^{2/3} \left[- \left(\frac{3dt}{R} + \frac{3}{2} (\cosh \rho - 1) d\theta^1 \right)^2 + \frac{3}{2} (d\rho^2 + \sinh^2 \rho (d\theta^1)^2) \right. \\ \left. + d\eta^2 + \sin^2 \eta d \left(\theta^2 + \frac{3}{2} \theta^1 - \frac{3t}{R} \right)^2 \right] \quad (4.74)$$

$$F^I = - \frac{p^I P}{2} \sin \eta d\eta \wedge d \left(\theta^2 + \frac{3}{2} \theta^1 - \frac{3t}{R} \right), \quad Y^1 = \left(\frac{6}{p^3} \right)^{\frac{1}{3}} p^I \quad (4.75)$$

As promised in section 3.3.4, this is indeed a ‘bubbled’ solution with a nontrivial S^2 supported by flux.

A first unpleasant property of the metric is that it has CTCs. Although the condition (4.72) removes CTCs that would otherwise appear near $\rho = 0$, the full solution develops CTCs for larger values of ρ since

$$g_{\theta^1 \theta^1} \sim \sinh^2 \frac{\rho}{2} (5 - \cosh \rho) + \frac{3 \sin^2 \eta}{2} \quad (4.76)$$

Comparing the first line of (4.75) with the global AdS_3 solution (4.43), we see that the timelike fiber is stretched. In fact, timelike stretched AdS_3 is the most well-known solution with CTCs, namely the Gödel universe [48].¹² Hence our solution represents the 3D Gödel universe with a spinning S^2 fibered over it.

Another property which makes the solution unlikely to represent the backreaction of a single brane is that it has too much symmetry: the symmetry is $\text{SL}(2, \mathbb{R}) \times \text{U}(1) \times \text{SO}(3)$ while from the probe analysis we learned that the brane should preserve only $\text{U}(1) \times \text{U}(1) \times \text{SO}(3)$. So does this solution actually contain any elliptic branes? At a first glance one might think that the answer is no: for $q = \frac{1}{2}$, the accumulated $\text{SL}(2, \mathbb{Z})$ monodromy under $\theta^1 \rightarrow \theta^1 + 2\pi$ is **-1**, i.e. the center of $\text{SL}(2, \mathbb{Z})$. Since τ itself doesn't feel the center of $\text{SL}(2, \mathbb{Z})$ and has no monodromy one might be tempted to conclude that for $q = \frac{1}{2}$ there is no actual brane source present. This conclusion would be wrong however, for the same reason that it would be in F-theory, because there are fields in the theory which do feel the center of $\text{SL}(2, \mathbb{Z})$ and have a monodromy. In the F-theory context, these fields are the NSNS and RR two forms and the object which produces the **-1** monodromy is a bound state of

¹²To be precise, Gödel's 4D solution was the product of 3D timelike stretched AdS_3 with a line [52].

an O7 plane and 4 D7 branes [53], while in the present context one can see from (2.20) that the two-forms Φ^+ and Φ^- transform under the center of $SL(2, \mathbb{Z})$ under which they pick up a sign. Hence we conclude that the Gödel $\times S^2$ solution contains a brane source in the origin. This is not the only brane source present however since, because the Gödel universe is spatially homogeneous, we could have actually made the previous argument for any point in the Gödel universe. In particular, the solution also contains brane sources on the boundary of the Gödel universe, which was already observed in [9].

Hence we conclude that the Gödel $\times S^2$ solution arises from filling $AdS_3 \times S^2$ with a congruence of $q = \frac{1}{2}$ elliptic branes, each of them wrapped on the S^2 and moving on a rotating worldline as in figure 3. Indeed, it is well known that the 3D Gödel universe arises from filling AdS_3 with rotating dust, and in the present supersymmetric context the rotating dust consists of $q = \frac{1}{2}$ elliptic branes. Indeed, one can show that the stress tensor of the axidilaton is precisely that of rotating dust [9].

What if we had started, instead of (4.65), from the source μ appropriate for an M2 or hyperbolic brane? As was discussed in section 3.3.4, the solutions in these classes IP, IH are in this case actually related by a holomorphic coordinate transformation of w^1 : they simply correspond to choosing ∂_{θ^1} to generate a parabolic or hyperbolic orbit on the hyperbolic plane instead of an elliptic one.

We also want to point out an interesting difference between this solution and the brane solutions in flat space of example 4.2.1. The latter were not globally consistent as τ_2 became negative in some region. This is not the case in the current example, since the locus $\rho \rightarrow \infty$ where τ_2 becomes zero coincides with the boundary of the Gödel spacetime.

4.2.5 Outlook: branes in curved backgrounds and black hole deconstruction

Having established that the Gödel $\times S^2$ solution does not describe the backreaction of a single brane but rather of a distribution of brane charges, we now return to the discussion of the backreacted solution of a single M2-brane in the Taub-NUT-anti-Taub-NUT background which we initiated in section 4.2.3. This solution is expected to be of the form (3.133)–(3.134) and is completely determined in terms of a function Φ obeying the nonlinear ordinary differential equation

$$\Phi'' + \frac{4}{P^2}(1 - px^1)e^{-2\Phi} = 0. \quad (4.77)$$

It should obey the subsidiary condition

$$\lim_{p \rightarrow 0} e^{\Phi} = 2 \sinh \frac{x^1}{P} \quad (4.78)$$

which expresses that we should recover the TN-anti-TN background when we turn off the M2 charge. The special solution (4.64) to (4.77), $e^{2\Phi} = \frac{8(1-px^1)^3}{3p^2P^2}$, which gives rise to the Gödel $\times S^2$ spacetime, doesn't satisfy the subsidiary condition as it becomes singular in the limit $p \rightarrow 0$. Hence if the desired solution is to be found in this class, there should exist more general solutions to (4.77). Naively we indeed expect the general solution to the second-order ODE (4.77) to depend on two integration constants, which have been fixed

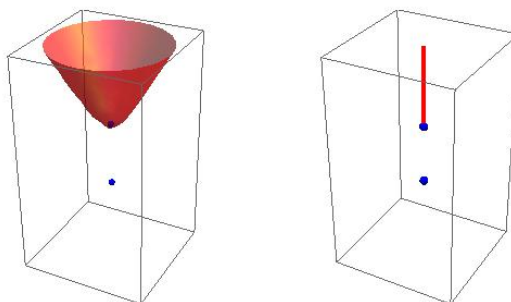


Figure 5. (a) From the 4D point of view, a brane probe at constant w^1 in the $\mathbb{R} \times \text{Eguchi-Hanson}$ background comes from a D2-brane on a single sheet of a hyperboloid in the D6-D6 background. (b) The collapsed D2 brane at $w^1 = -\infty$.

to specific values in (4.64). In support of this we show in appendix C that the solution to (4.77) on an interval is fixed uniquely once the values of Φ at the two endpoints are given. Another encouraging fact is that, as we saw in section (3.2.3), the generic solution to (4.77) leads to a metric with $U(1) \times U(1) \times SO(3)$ symmetry, which agrees with the symmetries (4.62) preserved by the M2-brane probe. Unfortunately, the general solution to (4.77) is not known analytically so that in order to make further progress we must resort to approximate methods to construct solutions obeying (4.77) and the subsidiary condition (4.78). We will discuss such methods and the physical properties of the resulting solution in a forthcoming publication [12].

Let's end by commenting on how similar methods are expected to lead to backreacted M2 brane solutions in the $\mathbb{R} \times \text{Eguchi-Hanson}$ and $\text{AdS}_2 \times S^3$ backgrounds. In the $\mathbb{R} \times \text{Eguchi-Hanson}$ background discussed in example 4.1.2, we can place an M2-brane on the holomorphic surface $w^1 = \text{constant}$, which describes the lift to 5D of a D2 brane on a single sheet of a hyperboloid in the presence of two D6-branes, see figure 5(a). The limiting case $w^1 = -\infty$ is the fixed locus of ∂_{θ^1} and inserting the brane there preserves toricity of the base. In the IIA picture, the D2 brane is collapsed and ends on one of the D6 branes, see figure 5(b). The backreacted solution is of the type (3.132) with $s_1 = 0, s_2 = 2, \kappa^2 = -4/P^2$ and with $\Phi(x^1)$ a solution of

$$\Phi'' - \frac{4}{P^2}(1 - px^1)e^{-2\Phi} = 0 \quad (4.79)$$

obeying the subsidiary condition

$$\lim_{p \rightarrow 0} e^{\Phi} = 2 \cosh \frac{x^1}{P}. \quad (4.80)$$

The same solution to (4.79), (4.80) also serves to describe a backreacted M2 brane located at $w^1 = -\infty$ in the $\text{AdS}_2 \times S^3$ background. In this case the M2 fills AdS_2 while tracing out a circle on the S^3 . The 5D solution is now of the form (3.133)–(3.134) with $s_1 = 0, s_2 = 2, \kappa^2 = 4/P^2, a^2 > 0, b = R = 0$.

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A Universal hypermultiplet target space

In this appendix we collect some formulae related to the universal hypermultiplet which will be of use in the main text. We refer to [18] for more details. The target space of the universal hypermultiplet is the 4 real-dimensional quaternionic space $\frac{\text{SU}(1,2)}{\text{U}(2)}$ on which we choose real coordinates $q^X, X = 1, \dots, 4$. In terms of the complex fields S, C

$$S = q_1 + iq_2 \quad (\text{A.1})$$

$$C = q_3 + iq_4 \quad (\text{A.2})$$

the metric reads

$$ds^2 = \psi^1 \bar{\psi}^1 + \psi^2 \bar{\psi}^2 \quad (\text{A.3})$$

with

$$\psi^1 = e^F dC \quad (\text{A.4})$$

$$\psi^2 = e^{2F} \left(\frac{dS}{2} - \bar{C} d\bar{C} \right) \quad (\text{A.5})$$

$$F = -\frac{1}{2} \log \left(\frac{1}{2} (S + \bar{S}) - C \bar{C} \right) \quad (\text{A.6})$$

The quaternionic structure $J_X^i, i = 1, \dots, 3$ is explicitly given by

$$\begin{aligned} J^1 &= \frac{1}{\sqrt{q_1 - q_3^2 - q_4^2}} \begin{pmatrix} q_4 & q_3 & 0 & \frac{1}{2} \\ -q_3 & q_4 & -\frac{1}{2} & 0 \\ -4q_3q_4 & 2(q_1 - 2q_3^2) & -q_4 & -q_3 \\ -2(q_1 - 2q_3^2) & -4q_3q_4 & q_3 & -q_4 \end{pmatrix} \\ J^2 &= \frac{1}{\sqrt{q_1 - q_3^2 - q_4^2}} \begin{pmatrix} q_3 & -q_4 & \frac{1}{2} & 0 \\ q_4 & q_3 & 0 & \frac{1}{2} \\ -2(q_1 - 2q_4^2) & 4q_3q_4 & -q_3 & q_4 \\ -4q_3q_4 & -2(q_1 - 2q_4^2) & -q_4 & -q_3 \end{pmatrix} \\ J^3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 4q_4 & 4q_3 & 0 & 1 \\ -4q_3 & 4q_4 & -1 & 0 \end{pmatrix} \end{aligned} \quad (\text{A.7})$$

These satisfy $(J^1)^2 = (J^2)^2 = (J^3)^2 = J^1 J^2 J^3 = -1$.

The spin connection splits in $SU(2) \times SU(2)'$ parts:

$$\omega = -\frac{i}{2}A^a\sigma^a \otimes \mathbf{1} + \mathbf{1} \otimes B^a\sigma^a,$$

with

$$\begin{aligned} A^1 &= \frac{2dq_4}{\sqrt{q_1 - q_3^2 - q_4^2}} \\ A^2 &= \frac{2dq_3}{\sqrt{q_1 - q_3^2 - q_4^2}} \\ A^3 &= -\frac{dq_2 + 2q_4dq_3 - 2q_3dq_4}{2(q_1 - q_3^2 - q_4^2)}. \end{aligned} \tag{A.8}$$

B Solving the vector multiplet equations

In this appendix, we derive the general form of solutions to the equations (2.8)–(2.10) determining how time is fibered over the 4D base as well the fields in the vector multiplets, in the case that there is a Killing vector under which τ is invariant. This means we have to solve (2.8)–(2.10) on a 4D base manifold with metric (3.21).

We can treat this system of equations as in the analysis of [22] which easily generalizes to the case when the axion-dilaton is turned on. To solve the first equation (2.8), we start from the ansatz

$$\Theta^I = \left(-2K^0 \star_3 d \left(\frac{K^I}{K^0} \right) \right)^- \tag{B.1}$$

$$= d \left(\frac{K^I}{K^0} (d\theta^2 + \chi) \right) - \star_3 \left(dK^I + \tilde{\tau}_2 e^{\tilde{h}_1} s_2 e^{s_2 \Psi} K^0 K^I dy_2 \right). \tag{B.2}$$

Using (3.40) and the property that, for a one-form α ,

$$\star_4 (\star_3 \alpha) = \alpha \wedge \frac{1}{K^0} (d\theta^2 + \chi) \tag{B.3}$$

one finds that demanding that Θ^I is closed is equivalent to

$$\nabla_3^2 K^I = -s_2 e^{-s_2 \Psi} \partial_{y_2} (K^0 K^I e^{s_2 \Psi}) \tag{B.4}$$

where the subscript $_3$ means that the covariant derivative is taken with respect to the 3D metric (3.22). For later use we note that closed selfdual forms can be constructed starting from the ansatz $\tilde{\Theta} = (2K^0 \star_3 dF)^+$, leading to the equation

$$\nabla_3^2 F = s_2 K^0 F_{y_2} \tag{B.5}$$

Defining $Z_I = f^{-1} Y_I$, the second equation (2.9) is equivalent to

$$\nabla_3^2 Z_I = 2D_{IJK} \nabla_3^i \left(\frac{K^J}{K^0} \right) \nabla_{3i} \left(\frac{K^K}{K^0} \right). \tag{B.6}$$

Plugging in the ansatz

$$Z_I = -2K_I + D_{IJK} \frac{K^J K^K}{K^0} \quad (\text{B.7})$$

and using (3.18), (B.4), leads to the following equation for K_I :

$$\nabla_3^2 K_I = -\frac{s_2 D_{IJK}}{2} e^{-s_2 \Psi} \partial_{y_2} (K^0 K^I e^{s_2 \Psi}) \quad (\text{B.8})$$

Turning to the last equation (2.10), we decompose ξ as

$$\xi = \nu(d\theta^2 + \chi) + \frac{\omega}{2} \quad (\text{B.9})$$

with ω a one-form on the 3D base. Contracting the equation with ∂_{θ^2} gives the equation for ω :

$$\star_3 d\omega = -2\nu dK^0 + 2K^0 d\nu - K^0 Z_I d\left(\frac{K^I}{K^0}\right) - 2s_2 \nu (K^0)^2 dy_2. \quad (\text{B.10})$$

The remaining equations are equivalent to the integrability condition for this equation and lead to

$$\nabla_3^2 \nu = \frac{1}{2K^0} \nabla_3^i \left(K^0 Z_I \partial_i \left(\frac{K^I}{K^0} \right) \right) + s_2 K^0 \partial_{y_2} \nu. \quad (\text{B.11})$$

Plugging in the ansatz

$$\nu = -\frac{K_I K^I}{2K^0} + D_{IJK} \frac{K^I K^J K^K}{6(K^0)^2} + \frac{K_0}{2} \quad (\text{B.12})$$

leads to the following equation for K_0 :

$$\nabla_3^2 K_0 = s_2 K^0 \partial_{y_2} K_0 - s_2 K^I \partial_{y_2} K_I - \frac{s_2^2}{6} D_{IJK} K^I K^J K^K. \quad (\text{B.13})$$

As a check we see from (B.5) that the freedom of adding a closed selfdual part to $d\xi$ corresponds to shifting K_0 by a solution of the homogeneous equation. Putting this all together leads to the general form of the solution (3.25)–(3.30) and the differential equations (3.36)–(3.41).

C Boundary value problem for the deformed Liouville equation

Here we will show, following [40], that the solution of the deformed Liouville equation (3.55) with $\kappa^2 < 0$ on an interval is uniquely determined by specifying its values at the endpoints. Setting $\kappa^2 = -1$, $s_2 = 0$ by redefining μ , the deformed Liouville equation (3.125) reads

$$\Phi'' + e^{\mu-2\Phi} = 0 \quad (\text{C.1})$$

which we consider on some interval \mathcal{D} of the real line. Suppose we have two solutions Φ_1, Φ_2 , then it follows that their difference $w = \Phi_1 - \Phi_2$ satisfies

$$w'' = e^{\mu-2\Phi_1} w (e^{2w} - 1) \geq 0 \quad (\text{C.2})$$

Multiplying with w and integrating we get

$$-\int_{\mathcal{D}} dx (w')^2 + (ww')|_{\partial\mathcal{D}} = \int_{\mathcal{D}} e^{\mu-2\Phi_1} w (e^{2w} - 1) \geq 0 \quad (\text{C.3})$$

If the boundary term vanishes, i.e. if we fix either Φ or Φ' on the boundary, we can conclude that $w = 0$ and the solution is unique. When the boundary behaviour is not fixed the solution depends on two integration constants as one would naively expect.

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References

- [1] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [[hep-th/9711053](#)] [[INSPIRE](#)].
- [2] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five-dimensions*, *Class. Quant. Grav.* **20** (2003) 4587 [[hep-th/0209114](#)] [[INSPIRE](#)].
- [3] J. Bellorín, P. Meessen and T. Ortín, *All the supersymmetric solutions of $N = 1$, $d = 5$ ungauged supergravity*, *JHEP* **01** (2007) 020 [[hep-th/0610196](#)] [[INSPIRE](#)].
- [4] M. Gutperle and M. Spalinski, *Supergravity instantons and the universal hypermultiplet*, *JHEP* **06** (2000) 037 [[hep-th/0005068](#)] [[INSPIRE](#)].
- [5] M. Gutperle and M. Spalinski, *Supergravity instantons for $N = 2$ hypermultiplets*, *Nucl. Phys. B* **598** (2001) 509 [[hep-th/0010192](#)] [[INSPIRE](#)].
- [6] M.H. Emam, *Wrapped M5-branes leading to five dimensional 2-branes*, *Phys. Rev. D* **74** (2006) 125004 [[hep-th/0610161](#)] [[INSPIRE](#)].
- [7] M.H. Emam, *Five dimensional 2-branes and the universal hypermultiplet*, *Nucl. Phys. B* **817** (2009) 208 [[hep-th/0701060](#)] [[INSPIRE](#)].
- [8] F. Denef, D. Gaiotto, A. Strominger, D. Van den Bleeken and X. Yin, *Black hole deconstruction*, *JHEP* **03** (2012) 071 [[hep-th/0703252](#)] [[INSPIRE](#)].
- [9] T.S. Levi, J. Raeymaekers, D. Van den Bleeken, W. Van Herck and B. Vercnocke, *Godel space from wrapped M2-branes*, *JHEP* **01** (2010) 082 [[arXiv:0909.4081](#)] [[INSPIRE](#)].
- [10] C.P. Boyer and I. Finley, J. D., *Killing vectors in selfdual, euclidean Einstein spaces*, *J. Math. Phys.* **23** (1982) 1126 [[INSPIRE](#)].
- [11] J.D. Gegenberg and A. Das, *Stationary riemannian space-times with self-dual curvature*, *Gen. Rel. Grav.* **16** (1984) 817.
- [12] J. Raeymaekers and D. Van den Bleeken, to appear.
- [13] M. Günaydin, G. Sierra and P.K. Townsend, *The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras*, *Nucl. Phys. B* **242** (1984) 244 [[INSPIRE](#)].
- [14] B. de Wit and A. Van Proeyen, *Broken σ -model isometries in very special geometry*, *Phys. Lett. B* **293** (1992) 94 [[hep-th/9207091](#)] [[INSPIRE](#)].

- [15] J. Bagger and E. Witten, *Matter couplings in $N = 2$ supergravity*, *Nucl. Phys. B* **222** (1983) 1 [INSPIRE].
- [16] A.C. Cadavid, A. Ceresole, R. D'Auria and S. Ferrara, *Eleven-dimensional supergravity compactified on Calabi-Yau threefolds*, *Phys. Lett. B* **357** (1995) 76 [hep-th/9506144] [INSPIRE].
- [17] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys. A* **4** (1989) 2475 [INSPIRE].
- [18] A. Strominger, *Loop corrections to the universal hypermultiplet*, *Phys. Lett. B* **421** (1998) 139 [hep-th/9706195] [INSPIRE].
- [19] A.V. Pogorelov, *Extrinsic geometry of convex surfaces*, Translations of Mathematical Monographs, American Mathematical Society, U.S.A. (1973).
- [20] M.V. Saveliev, *On the integrability problem of a continuous Toda system*, *Theor. Math. Phys.* **92** (1992) 457.
- [21] M.V. Saveliev and S.A. Saveleva, *$W(\infty)$ geometry and associated continuous Toda system*, *Phys. Lett. B* **313** (1993) 55 [hep-th/9305152] [INSPIRE].
- [22] I. Bena, N. Bobev and N.P. Warner, *Bubbles on manifolds with a $U(1)$ isometry*, *JHEP* **08** (2007) 004 [arXiv:0705.3641] [INSPIRE].
- [23] D. Gaiotto, A. Strominger and X. Yin, *5D black rings and 4D black holes*, *JHEP* **02** (2006) 023 [hep-th/0504126] [INSPIRE].
- [24] J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, *A Farey tail for attractor black holes*, *JHEP* **11** (2006) 024 [hep-th/0608059] [INSPIRE].
- [25] M.C.N. Cheng, *More bubbling solutions*, *JHEP* **03** (2007) 070 [hep-th/0611156] [INSPIRE].
- [26] I. Bena, P. Kraus and N.P. Warner, *Black rings in Taub-NUT*, *Phys. Rev. D* **72** (2005) 084019 [hep-th/0504142] [INSPIRE].
- [27] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049] [INSPIRE].
- [28] I. Bena and N.P. Warner, *Bubbling supertubes and foaming black holes*, *Phys. Rev. D* **74** (2006) 066001 [hep-th/0505166] [INSPIRE].
- [29] K.P. Tod, *All metrics admitting supercovariantly constant spinors*, *Phys. Lett. B* **121** (1983) 241 [INSPIRE].
- [30] K.P. Tod, *More on supercovariantly constant spinors*, *Class. Quant. Grav.* **12** (1995) 1801 [INSPIRE].
- [31] P. Meessen and T. Ortín, *The supersymmetric configurations of $N = 2$, $D = 4$ supergravity coupled to vector supermultiplets*, *Nucl. Phys. B* **749** (2006) 291 [hep-th/0603099] [INSPIRE].
- [32] G.W. Gibbons and S.W. Hawking, *Classification of gravitational instanton symmetries*, *Commun. Math. Phys.* **66** (1979) 291 [INSPIRE].
- [33] D. Gaiotto, A. Strominger and X. Yin, *New connections between 4D and 5D black holes*, *JHEP* **02** (2006) 024 [hep-th/0503217] [INSPIRE].
- [34] K. Behrndt, G. Lopes Cardoso and S. Mahapatra, *Exploring the relation between 4D and 5D BPS solutions*, *Nucl. Phys. B* **732** (2006) 200 [hep-th/0506251] [INSPIRE].

- [35] J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. Van den Bleeken, *Black hole bound states in $AdS_3 \times S^2$* , *JHEP* **11** (2008) 050 [[arXiv:0802.2257](#)] [[INSPIRE](#)].
- [36] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, *JHEP* **11** (2011) 127 [[hep-th/0304094](#)] [[INSPIRE](#)].
- [37] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, *JHEP* **11** (2011) 129 [[hep-th/0702146](#)] [[INSPIRE](#)].
- [38] I. Bena and N.P. Warner, *Black holes, black rings and their microstates*, *Lect. Notes Phys.* **755** (2008) 1 [[hep-th/0701216](#)] [[INSPIRE](#)].
- [39] M. Kleban and M. Redi, *Expanding F-theory*, *JHEP* **09** (2007) 038 [[arXiv:0705.2020](#)] [[INSPIRE](#)].
- [40] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, *Trans. Amer. Math. Soc.* **324** (1991) 793.
- [41] M. Abreu and R. Sena-Dias, *Scalar-flat Kahler metrics on non-compact symplectic toric 4-manifolds*, [arXiv:0910.5466](#).
- [42] A. Weinstein, *Generalized axially symmetric potential theory*, *Bull. Amer. Math. Soc.* **59** (1953) 20.
- [43] J. de Boer and M. Shigemori, *Exotic branes and non-geometric backgrounds*, *Phys. Rev. Lett.* **104** (2010) 251603 [[arXiv:1004.2521](#)] [[INSPIRE](#)].
- [44] J. de Boer and M. Shigemori, *Exotic branes in string theory*, *Phys. Rept.* **532** (2013) 65 [[arXiv:1209.6056](#)] [[INSPIRE](#)].
- [45] B.R. Greene, A.D. Shapere, C. Vafa and S.-T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nucl. Phys. B* **337** (1990) 1 [[INSPIRE](#)].
- [46] E.A. Bergshoeff, J. Hartong, T. Ortín and D. Roest, *Seven-branes and supersymmetry*, *JHEP* **02** (2007) 003 [[hep-th/0612072](#)] [[INSPIRE](#)].
- [47] A.P. Braun, A. Hebecker and H. Triendl, *D7-brane motion from M-theory cycles and obstructions in the weak coupling limit*, *Nucl. Phys. B* **800** (2008) 298 [[arXiv:0801.2163](#)] [[INSPIRE](#)].
- [48] K. Gödel, *An example of a new type of cosmological solutions of Einstein's field equations of gravitation*, *Rev. Mod. Phys.* **21** (1949) 447 [[INSPIRE](#)].
- [49] T. Eguchi and A.J. Hanson, *Asymptotically flat selfdual solutions to euclidean gravity*, *Phys. Lett. B* **74** (1978) 249 [[INSPIRE](#)].
- [50] T. Eguchi and A.J. Hanson, *Selfdual solutions to euclidean gravity*, *Annals Phys.* **120** (1979) 82 [[INSPIRE](#)].
- [51] D. Gaiotto, A. Strominger and X. Yin, *Superconformal black hole quantum mechanics*, *JHEP* **11** (2005) 017 [[hep-th/0412322](#)] [[INSPIRE](#)].
- [52] M. Rooman and P. Spindel, *Gödel metric as a squashed Anti-de Sitter geometry*, *Class. Quant. Grav.* **15** (1998) 3241 [[gr-qc/9804027](#)] [[INSPIRE](#)].
- [53] A. Sen, *F theory and orientifolds*, *Nucl. Phys. B* **475** (1996) 562 [[hep-th/9605150](#)] [[INSPIRE](#)].